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AC properties of 3d Sierpinski gaskets: rigorous results

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Abstract

The problem of the decimation of a network of impedances on the three-dimensional Sierpinski gasket is solved: the exact map $\mathcal{M}$ is given and its asymptotic behaviours are studied. The most significant invariant subspaces of $\mathcal{M}$ and the associated submaps are considered. This also allows us to address the problem of small-size phenomena, such as oscillating asymptotic behaviour, on this kind of fractal. The set of the resonances of the system and the frequency dependence of the total impedance are studied both in the thermodynamic limit and in mesoscopic systems.

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1. Introduction

In recent years we have been witnessing a growing rate of interest in the practical application of fractals. Their peculiar geometry was used, for instance, in optics to study the properties of fractal diffraction gratings (diffractals) [1] and to create subwavelength fractal bandgap slabs [2]. In electrical engineering, electromagnetic theory has been combined with fractal geometry to found a new branch of research called *fractal electrodynamics* [3], in which the most promising area is the theory and design of multiband fractal antennas [4]. As opposed to the usual statistical models on fractals, that are usually studied in the thermodynamic limit, all these systems are typically low size; nevertheless their self-similarity has a profound influence on their physical properties.

In a previous paper [5] we addressed a simple electrical problem on a fractal: that of a deterministic circuit of passive impedances (resistors, inductors and capacitors), choosing the paradigmatic Sierpinski gasket to be our system. We detected typical signs of the fractal geometry of the system, e.g., in the distribution of electrical resonances and in the dependence of the total impedance on the frequency of the applied signal. We also found a signature

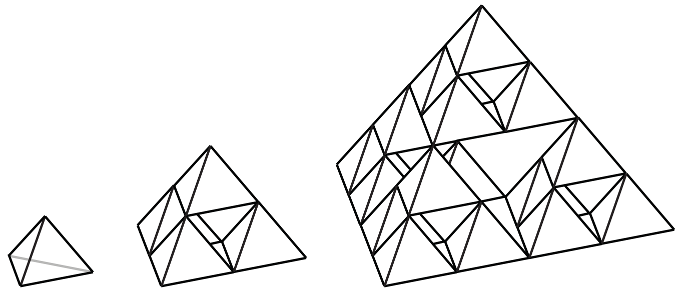

Figure 1. The basic cell (generation 0) and the first two generations of the three-dimensional Sierpinski gasket.

of the dilatation-invariant structure in a phenomenon, which we called oscillating asymptotic behaviour, that was peculiar of mesoscopic systems.

In this paper we extend our analysis: we find and discuss the exact recursion map for the decimation of a 3d Sierpinki gasket of impedances. This problem was first tackled 20 years ago by Vannimenus and Knezevich [6], who calculated the asymptotic expansion of the map to first order: they aimed at modelling the isotropization of macroscopic conductivity near the percolation threshold in anisotropic systems. Later on, Jafarizadeh [7], motivated by the results of Barlow *et al* [8], gave a general method of finding the map of any d-dimensional generalized Sierpinski gasket, and found the exponents with which the anisotropy vanishes.

The exact map that we will deduce in the following will allow us to classify the fixed points (apart from that at infinity) and identify several invariant subspaces and submaps with exotic behaviour. We will be able to study small-size effects, such as oscillating asymptotic behaviour (OAB), and the dependence on frequency both at low and high generations.

The outline of the paper is as follows. In section 2 we show how to find the exact map $\mathcal{M}$ (that is given in the appendix) and study its fixed points and asymptotic behaviours. In section 3 we consider the most interesting invariant subspaces of $\mathcal{M}$. In section 4 we introduce and analyse the oscillating asymptotic behaviour. In section 5 we address the issue of the frequency dependence of the impedance of our system in the most interesting cases. Section 6 is devoted to the conclusions and the perspectives for future work.

2. The exact map

2.1. Calculation of the map

The network we consider is a 3d Sierpinski gasket of impedances: the recursive rule to build it is shown in figure 1.

The basic cell (figure 2) is a tetrahedron with four vertices and six links, an electrical pole on each vertex and an impedance on each link; it coincides with the complete graph K_4, as shown on the right.

The vertices are labelled with the lower case letters a, b, c, d and the links with the numbers $1, \ldots, 6$ as shown in the figure; accordingly, the impedances on the links are z_i, $i = 1, \ldots, 6$. When necessary we will use the notation $z_i^{(n)}, i = 1, \ldots, 6$ to indicate nth-generation impedances (i.e., the impedances after n decimations). We also call $\mathcal{Z}_{ab}$ the external impedance of the system measured between the two poles a and b ($\mathcal{Z}_{ab}^{(n)}$ at generation n), and the same for the other poles (there are six ways to measure the external impedance).

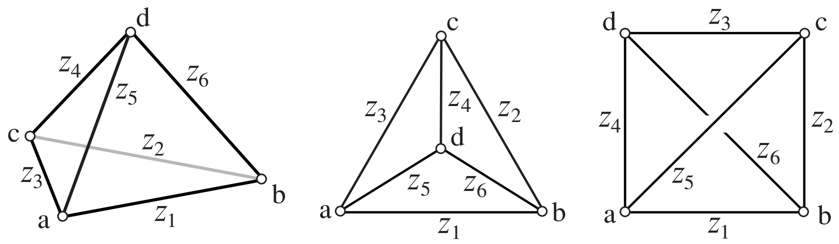

Figure 2. Left: the basic cell of the fractal is a tetrahedron with an impedance on each link. Middle: the same viewed from above. Right: the basic cell is the complete graph K_4, where each pair of vertices is connected by a link.

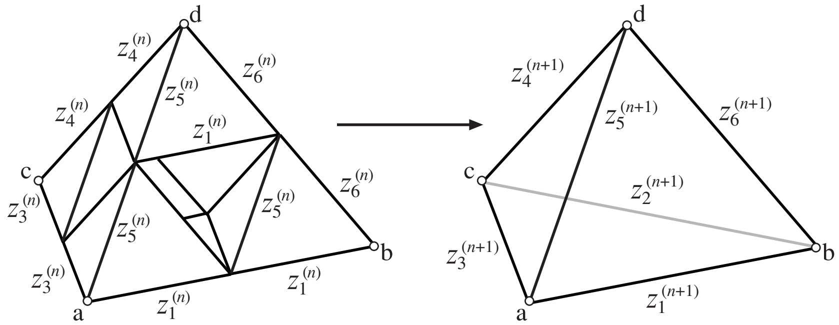

Figure 3. Decimation procedure for the gasket of impedances, from generation n (left) to generation $n + 1$ (right).

Our purpose is to find the exact renormalization map connecting the impedances of generation n, $z_i^{(n)}$, to those of generation $n + 1$, $z_i^{(n+1)}$ (figure 3). In the 2d case [5] this is an easy task thanks to the existence of the so-called star-triangle (or $Y - \Delta$) transformations, that allow one to eliminate a loop from the circuit by adding a vertex and vice-versa, but such methods do not work in three dimensions. Therefore an alternative approach is required.

The simplest way is to follow the method used in [6] and write down the equations for the potentials and currents in the two systems according to Kirchhoff's laws. In order to do so we switch to conductances: we will revert to impedances at the end of the calculation. Let

$$\sigma_i = 1\big/z_i^{(n)} \qquad \Sigma_i = 1\big/z_i^{(n+1)} \tag{1}$$

be the conductances of link i ($i = 1, \ldots, 6$) at generation n and $n + 1$, respectively (figure 4). For both generations we call I_a, V_a the incoming current and the potential at pole a and the same for the other poles. We can put one pole to the ground, $V_a = 0$, and since the conservation law for currents states that $I_a = -I_b - I_c - I_d$ we are left with three potentials and three currents.

In matrix form the Kirchhoff equations for the nth-generation gasket read

$$\begin{pmatrix} I_b \\ I_c \\ I_d \end{pmatrix} = \begin{pmatrix} \Sigma_1 + \Sigma_2 + \Sigma_6 & -\Sigma_2 & -\Sigma_6 \\ -\Sigma_2 & \Sigma_2 + \Sigma_3 + \Sigma_4 & -\Sigma_4 \\ -\Sigma_6 & -\Sigma_4 & \Sigma_4 + \Sigma_5 + \Sigma_6 \end{pmatrix} \begin{pmatrix} V_b \\ V_c \\ V_d \end{pmatrix}. \tag{2}$$

For the sake of brevity we call $[\Sigma]$ the 3×3 matrix of the $(n + 1)$th-order conductances, and $\vec{I}$, $\vec{V}$ the vectors of the external currents and potentials, so that the equations take on the compact form $\vec{I} = [\Sigma] \cdot \vec{V}$.

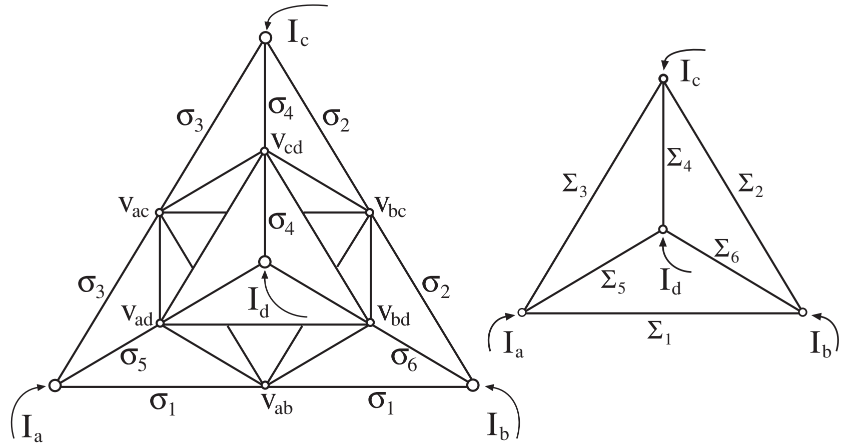

Figure 4. We call σ_i the conductances of the *n*th-generation gasket (left) and Σ_i those of the $(n+1)$th-generation gasket (right). I_a and V_a are the incoming current and the potential at pole *a*, and so on for all the poles. In the *n*th-generation gasket v_{ab} is the potential at the middle point between poles *a* and *b*, and so on.

In the *n*th generation we call v_{ab} the potential at the middle point between pole *a* and pole *b* and the same for all other internal points (there are six such points; see figure 4). The equations for the *n*th-generation gasket are nine, namely, the following six for the inner poles:

$$[\sigma]\cdot\vec{v} = \begin{pmatrix} \sigma_1 V_b \\ \sigma_2(V_b+V_c) \\ \sigma_3 V_c \\ \sigma_4(V_c+V_d) \\ \sigma_5 V_d \\ \sigma_6(V_b+V_d) \end{pmatrix}; \qquad \vec{v} = \begin{pmatrix} v_{ab} \\ v_{bc} \\ v_{ac} \\ v_{cd} \\ v_{ad} \\ v_{bd} \end{pmatrix} \tag{3}$$

where $[\sigma]$ is the matrix of the *n*th-order conductances

$$[\sigma] = \begin{pmatrix} 2\sigma_1+\sigma_2+\sigma_3+\sigma_5+\sigma_6 & -\sigma_3 & -\sigma_2 & 0 & -\sigma_6 & -\sigma_5 \\ -\sigma_3 & \sigma_1+2\sigma_2+\sigma_3+\sigma_4+\sigma_6 & -\sigma_1 & -\sigma_6 & 0 & -\sigma_4 \\ -\sigma_2 & -\sigma_1 & \sigma_1+\sigma_2+2\sigma_3+\sigma_4+\sigma_5 & -\sigma_5 & -\sigma_4 & 0 \\ 0 & -\sigma_6 & -\sigma_5 & \sigma_2+\sigma_3+2\sigma_4+\sigma_5+\sigma_6 & -\sigma_3 & -\sigma_2 \\ -\sigma_6 & 0 & -\sigma_4 & -\sigma_3 & \sigma_1+\sigma_3+\sigma_4+2\sigma_5+\sigma_6 & -\sigma_1 \\ -\sigma_5 & -\sigma_4 & 0 & -\sigma_2 & -\sigma_1 & \sigma_1+\sigma_2+\sigma_4+\sigma_5+2\sigma_6 \end{pmatrix} \tag{4}$$

and three equations for the outer poles:

$$\begin{aligned} I_b &= (\sigma_1+\sigma_2+\sigma_6)V_b - \sigma_1 v_{ab} - \sigma_2 v_{bc} - \sigma_6 v_{bd} \\ I_c &= (\sigma_2+\sigma_3+\sigma_4)V_c - \sigma_2 v_{bc} - \sigma_3 v_{ac} - \sigma_4 v_{cd} \\ I_d &= (\sigma_4+\sigma_5+\sigma_6)V_d - \sigma_4 v_{cd} - \sigma_5 v_{ac} - \sigma_6 v_{bd}. \end{aligned} \tag{5}$$

We now proceed as follows. First, we solve equation (3) by inverting $[\sigma]$ and find $\vec{v}$ as a linear function of V_b, V_c, V_d and a rational function of the $\{\sigma_i\}$. Then we plug this solution into equation (4) to obtain a system of the form

$$\begin{pmatrix} I_b \\ I_c \\ I_d \end{pmatrix} = \begin{pmatrix} f_{11}(\{\sigma_i\}) & f_{12}(\{\sigma_i\}) & f_{13}(\{\sigma_i\}) \\ f_{12}(\{\sigma_i\}) & f_{22}(\{\sigma_i\}) & f_{23}(\{\sigma_i\}) \\ f_{13}(\{\sigma_i\}) & f_{23}(\{\sigma_i\}) & f_{33}(\{\sigma_i\}) \end{pmatrix} \begin{pmatrix} V_b \\ V_c \\ V_d \end{pmatrix} \tag{6}$$

where the f_{ij} are rational functions of degree 7 of the $\{\sigma_i\}$ that we will not write down here. The system has the same form as (2); thus we can write

$$\begin{aligned} &\Sigma_1 + \Sigma_2 + \Sigma_6 = f_{11}(\{\sigma_i\}) \\ &\Sigma_2 = -f_{12}(\{\sigma_i\}) \\ &\Sigma_6 = -f_{13}(\{\sigma_i\}) \\ &\Sigma_2 + \Sigma_3 + \Sigma_4 = f_{22}(\{\sigma_i\}) \\ &\Sigma_4 = -f_{23}(\{\sigma_i\}) \\ &\Sigma_4 + \Sigma_5 + \Sigma_6 = f_{11}(\{\sigma_i\}) \end{aligned} \tag{7}$$

and this system can be easily solved. We find that the Σ_i depend upon the σ_i through a rational map of degree 7:

$$\Sigma_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 \frac{P_6(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)}{Q_6(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)} \tag{8}$$

where P_6, Q_6 are homogeneous polynomials of degree 6 (the other Σ_i are obtained with suitable permutations: see below the considerations for the $z_i^{(n)}$). We now switch back to the impedance space by means of equations (1).

2.2. *Characteristics of the map*

In the impedance space the general solution takes on the form of a six-dimensional, homogeneous, rational map $\mathcal{M}$ of degree 13: $\left\{z_1^{(n+1)}, z_2^{(n+1)}, z_3^{(n+1)}, z_4^{(n+1)}, z_5^{(n+1)}, z_6^{(n+1)}\right\} = \mathcal{M}\left(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)}, z_5^{(n)}, z_6^{(n)}\right)$. The first equation of the map is

$$z_1^{(n+1)}\left(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)}, z_5^{(n)}, z_6^{(n)}\right) = z_1^{(n)} \frac{P_{12}\left(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)}, z_5^{(n)}, z_6^{(n)}\right)}{Q_{12}\left(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)}, z_5^{(n)}, z_6^{(n)}\right)} \tag{9}$$

where P_{12} and Q_{12} are homogeneous polynomials of degree 12 in the variables $\{z_i\}$; the polynomials are given in the appendix[1]. The other five equations are obtained from the first by means of a set of permutations that correspond to the elements of the point group of rotations of the tetrahedron [2]. These permutations are

$$\begin{aligned} &z_1 : (1, 2, 3, 4, 5, 6) \\ &z_2 : (2, 3, 1, 5, 6, 4) \\ &z_3 : (3, 2, 1, 6, 5, 4) \\ &z_4 : (4, 2, 6, 1, 5, 3) \\ &z_5 : (5, 3, 4, 2, 6, 1) \\ &z_6 : (6, 1, 5, 3, 4, 2). \end{aligned} \tag{10}$$

[1] Despite the higher degree, the polynomials in the variables $\{z_i\}$ are not less simple than those in the $\{\sigma_i\}$: indeed, there exists a one-to-one correspondence between the terms in (8) and those in (9). We choose to use the $\{z_i\}$ for consistency with our previous work [5].

[2] The group is indeed of order 12, but only six of its elements are relevant for our purposes, since permutations like $(1, 2, 3, 4, 5, 6) \to (1, 5, 6, 4, 2, 3)$ leave the map unchanged.

For example, at any generation $z_4^{(n+1)}(z_1, z_2, z_3, z_4, z_5, z_6) = z_1^{(n+1)}(z_4, z_2, z_6, z_1, z_5, z_3)$ and so on.

The map acts on the six-dimensional complex space: $\mathcal{M} : \hat{\mathbf{C}}^6 \to \hat{\mathbf{C}}^6$, where $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$; we refer the reader to the specific literature [9] for the language of dynamical systems and rational maps. $\mathcal{M}$ is invariant under the permutations (10) (that we shall refer to only as 'permutations' in the following). As a consequence, the properties of a particular 6-uple $\{z_i\}$ of impedances also hold for all its permutations. Also, the properties of a particular subspace (such as $\sum a_i z_i = 0$) are shared by all the subspaces obtained using these permutations (for example permuting the a_i). In such cases we will treat only one permutation without mentioning the other ones.

Another feature of the map $\mathcal{M}$ is that it is homogeneous of degree 1 in its variables: $\mathcal{M}(\{\lambda z_i\}) = \lambda \mathcal{M}(\{z_i\})$, $\lambda \in \mathbf{C}$. This property allows us to recover a physical meaning for those points with a negative real value (the physical constraint on the impedance Z of a passive element being that $\mathrm{Re}(Z) \geqslant 0$). In fact, if a result holds for a 6-uple $\{z_i\}$ it also holds for all the 6-uples $\{\lambda z_i\}$, $\lambda \in \mathbf{C}$. So a point is 'physically meaningful' provided that there exists a $\lambda \in \mathbf{C}$ such that $\{\mathrm{Re}(\lambda z_i) \geqslant 0\}$. For instance, a 6-uple of impedances such as $(-1, 1, 1, 1, 1, -1)$ makes sense since it can be multiplied, e.g., for i to give $(-i, i, i, i, i, -i)$ (a set of capacitive and inductive impedances), while the 6-uple $(1 + i, -1 + i, -i, 1 + i, -1 - i, -i)$ cannot be mapped by multiplication into any physically meaningful point. The requisite a set of impedances must satisfy to have a physical meaning is that their vectors in the complex plane cover an angle $\leqslant 180^\circ$.

A very important set is that of the backward orbit $\mathcal{O}$ of the point ∞, that is, the points that are poles of some iterate of the map: $\mathcal{O} = \{\{z_i\} : \mathcal{M}^n(\{z_i\}) = \infty$ for some $n\}$ (by this we mean that all the components of the iterate are infinite). They correspond to electrical resonances of the nth-generation gasket [3]. We distinguish the order of the iterate of the map by calling $\mathcal{O}_n$ the set of the points $\{z_i\}$ that are poles of the $(n+1)$th iterate: $\mathcal{O}_n = \{\{z_i\} : \mathcal{M}^{n+1}(\{z_i\}) = \infty\}$, so that $\mathcal{O} = \bigcup_{n=0}^{\infty} \mathcal{O}_n$. The $\mathcal{O}_n$ can be built recursively by calculating the preimages of the poles of $\mathcal{M}$: the resonances of the first-generation gasket are the points that make the denominator of $\mathcal{M}$ vanish: $\mathcal{M}(\mathcal{O}_0) = \infty$; their preimages are the resonances of the second-generation gasket: $\mathcal{M}(\mathcal{O}_1) = \mathcal{O}_0$, so that $\mathcal{M}^2(\mathcal{O}_1) = \infty$; and so on, with $\mathcal{M}(\mathcal{O}_{n+1}) = \mathcal{O}_n$ at each step. The $\mathcal{O}_n$ (and hence their union $\mathcal{O}$) are five-dimensional subvarieties of the space $\mathbf{C}^6$ and, due to the homogeneity of $\mathcal{M}$, they are generalized cones with axes lying on the line $z_1 = z_2 = z_3 = z_4 = z_5 = z_6$. We will examine a subset $\mathcal{C}$ of $\mathcal{O}$ below (par. III.C) and see that it is fractal.

The map is far too complex to be examined as it is: first we will study its fixed points and asymptotic behaviour, then we will examine some of its properties through its invariant subspaces.

2.3. Fixed points and asymptotic behaviour

We identify a set of impedances $\{z_1, z_2, z_3, z_4, z_5, z_6\}$ with the vector $\vec{v} = (z_1, z_2, z_3, z_4, z_5, z_6)$, $\vec{v} \in \mathbf{C}^6$.

The subspace $(z_1, z_2, z_3, -z_1, -z_2, -z_3)$ is a set of repelling fixed points of the map. The point to infinity $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = \infty$ is the only attracting fixed point of the map. The recursion relations can be linearized in the vicinity of this fixed point to yield

$$z_1^{(n+1)} = \tfrac{9}{8} z_1^{(n)} + \tfrac{1}{8} z_2^{(n)} + \tfrac{1}{8} z_3^{(n)} - \tfrac{1}{8} z_4^{(n)} + \tfrac{1}{8} z_5^{(n)} + \tfrac{1}{8} z_6^{(n)} \tag{11}$$

and permutations.

[3] Another set of electrical resonances, that we will not discuss here, is that of the points whose nth iterate is 0.

We give below the diagonalized variables, written as vectors in the $\{z_i\}$ space, together with their eigenvalues [4]:

$$
\begin{array}{ll}
\text{eigenvector} & \text{eigenvalue} \\
v_1 = (1, 2, -3, 1, 2, -3) & \frac{3}{4} \\
v_2 = (-5, 4, 1, -5, 4, 1) & \frac{3}{4} \\
v_3 = (1, 1, 1, -1, -1, -1) & \frac{5}{4} \\
v_4 = (3, -2, -1, -3, 2, 1) & \frac{5}{4} \\
v_5 = (1, 4, -5, -1, -4, 5) & \frac{5}{4} \\
v_6 = (1, 1, 1, 1, 1, 1) & \frac{3}{2}.
\end{array}
\tag{12}
$$

This result partitions the space $\hat{\mathbf{C}}^6$ into three orthogonal subspaces: we call $\mathcal{S}_{3/4}$ the subspace generated by vectors v_1, v_2; $\mathcal{S}_{5/4}$ that generated by v_3, v_4, v_5; $\mathcal{S}_{3/2}$ that generated by v_6. Thus, if we follow the evolution of a given point $\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6)$ and call $\vec{z}_{3/4}, \vec{z}_{5/4}$, and $z_{3/2}$ the projections of $\vec{z}$ onto $\mathcal{S}_{3/4}, \mathcal{S}_{5/4}$, and $\mathcal{S}_{3/2}$ respectively, the following asymptotic laws hold:

$$
\begin{aligned}
\left|\vec{z}_{3/4}^{(n)}\right| &\sim (3/4)^n \\
\left|\vec{z}_{5/4}^{(n)}\right| &\sim (5/4)^n \\
\left|z_{3/2}^{(n)}\right| &\sim (3/2)^n
\end{aligned}
\tag{13}
$$

(cf the results obtained in [6] with a first-order series expansion of the problem from the start). The evolution of $\vec{z}_{3/4}$ follows the first of equations (13) only for a few steps; then the first-order term $\sim(3/4)^n$ gets smaller than the second-order terms of the series expansion (that we do not report here) and the asymptotic law becomes $\left|\vec{z}_{3/4}^{(n)}\right| \sim (25/24)^n$.

3. Invariant subspaces

In this section we will examine some invariant subspaces of the map (keeping in mind that some of them can overlap); they are shown in figure 5.

*3.1. Subspace **I**:* $(z_1, z_1, z_1, z_4, z_4, z_4)$

This subspace is obtained when the three links on the basis of the tetrahedron have the same impedance z_1 and the three links going out from the vertex above have the same impedance z_4. It is generated by vectors v_3 and v_6; the external impedances are $\mathcal{Z}_{ab} = \mathcal{Z}_{ac} = \mathcal{Z}_{bc}$ and $\mathcal{Z}_{ad} = \mathcal{Z}_{bd} = \mathcal{Z}_{cd}$. The submap is

$$
z_1^{(n+1)} = \frac{2z_1^{(n)}z_4^{(n)}\left(z_1^{(n)} + 2z_4^{(n)}\right)\left(7z_1^{(n)} + 5z_4^{(n)}\right)}{\left(z_1^{(n)} + 3z_4^{(n)}\right)\left(\left(z_1^{(n)}\right)^2 + 7z_1^{(n)}z_4^{(n)} + 4\left(z_4^{(n)}\right)^2\right)}
$$

[4] We have avoided using basis vectors, like, e.g., (1, 0, 0, 1, 0, 0) for subspace $\mathcal{S}_{5/4}$, that would be simpler but belong to the set $\mathcal{O}_0$ of the poles of the map.

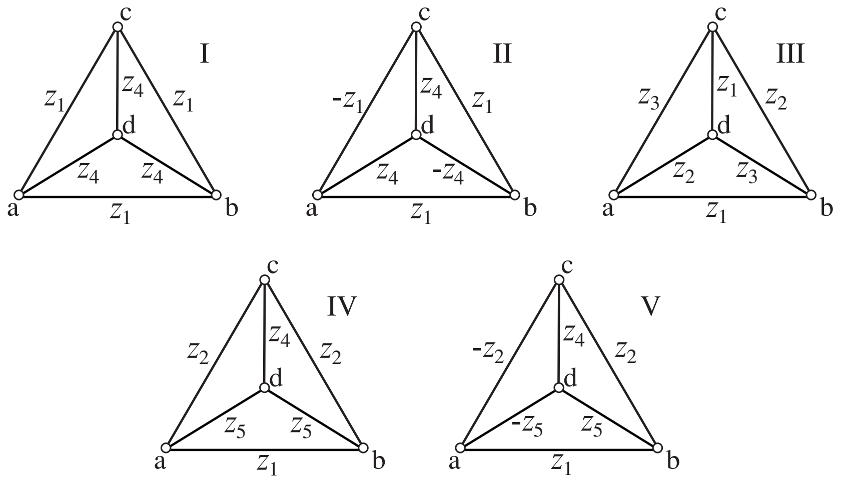

Figure 5. The subspaces we considered.

$$z_4^{(n+1)} = \frac{2z_4^{(n)}\left(z_1^{(n)} + 2z_4^{(n)}\right)}{z_1^{(n)} + 3z_4^{(n)}}. \tag{14}$$

We find that $z_1 = -z_4$ is a set of repelling fixed points, while $z_1 = z_4 = \infty$ is an attracting fixed point. The asymptotic laws for the latter are $\left|z_1^{(n)} - z_4^{(n)}\right| \sim (5/4)^n$, $\left|z_1^{(n)} + z_4^{(n)}\right| \sim (3/2)^n$.

3.2. *Subspace* ***II****:* $(z_1, -z_1, z_1, z_4, -z_4, z_4)$

The submap is

$$\begin{aligned} z_1^{(n+1)} &= \frac{-2z_1^{(n)}z_4^{(n)}\left(z_1^{(n)} + 2z_4^{(n)}\right)}{\left(z_1^{(n)}\right)^2 - z_1^{(n)}z_4^{(n)} - 4\left(z_4^{(n)}\right)^2} \\ z_4^{(n+1)} &= \frac{z_4^{(n)}\left(z_1^{(n)} + 2z_4^{(n)}\right)}{z_1^{(n)} + 3z_4^{(n)}}. \end{aligned} \tag{15}$$

The peculiarity of this submap is that the quantity $\tilde{z} = z_1^{(n)}z_4^{(n)}\big/\left(z_4^{(n)} - z_1^{(n)}\right)$ is constant, so that the whole orbit of the system lies on the surface of equation $z_1z_4/(z_4 - z_1) = z_1^{(0)}z_4^{(0)}\big/\left(z_4^{(0)} - z_1^{(0)}\right) = \tilde{z}$. The variable z_4 is marginal for the asymptotic linear expansion: the attracting fixed point of the map is indeed $(z_1, z_4) = (\tilde{z}, \infty)$, with the following asymptotic laws (that differ from the general ones):

$$\left|z_1^{(n)} - \tilde{z}\right| \sim (3/4)^n \qquad \left|z_4^{(n)}\right| \sim (4/3)^n. \tag{16}$$

The external impedances are $\mathcal{Z}_{ab} = \mathcal{Z}_{ac} = 0$; $\mathcal{Z}_{ac} = -\tilde{z}$; $\mathcal{Z}_{cd} = \mathcal{Z}_{bd} = z_4$, while $\mathcal{Z}_{ad} \sim z_4$ asymptotically. There are no repelling fixed points.

3.3. *Subspace* ***III****:* $(z_1, z_2, z_3, z_1, z_2, z_3)$

This subspace exploits a particular symmetry of the tetrahedron: we impose each pair of orthogonal links (in the K_4 graph, each pair of links that do not share a vertex) to have the same impedance. The subspace is generated by vectors v_1, v_2, v_6; the external impedances are $\mathcal{Z}_{ab} = \mathcal{Z}_{cd}, \mathcal{Z}_{bc} = \mathcal{Z}_{ad}, \mathcal{Z}_{ac} = \mathcal{Z}_{cd}$. The submap is

$$\begin{aligned} z_1^{(n+1)} &= 2z_1^{(n)} \frac{z_1^{(n)} z_2^{(n)} + z_1^{(n)} z_3^{(n)} + z_2^{(n)} z_3^{(n)}}{\left(z_1^{(n)} + z_2^{(n)}\right)\left(z_1^{(n)} + z_3^{(n)}\right)} \\ z_2^{(n+1)} &= 2z_2^{(n)} \frac{z_1^{(n)} z_2^{(n)} + z_1^{(n)} z_3^{(n)} + z_2^{(n)} z_3^{(n)}}{\left(z_1^{(n)} + z_2^{(n)}\right)\left(z_2^{(n)} + z_3^{(n)}\right)} \\ z_3^{(n+1)} &= 2z_3^{(n)} \frac{z_1^{(n)} z_2^{(n)} + z_1^{(n)} z_3^{(n)} + z_2^{(n)} z_3^{(n)}}{\left(z_1^{(n)} + z_3^{(n)}\right)\left(z_2^{(n)} + z_3^{(n)}\right)}. \end{aligned} \tag{17}$$

There are no finite fixed points (except points like $(0, z_2, z_2)$ that are poles of $\mathcal{M}$ and we do not consider). The attracting fixed point is $z_1 = z_2 = z_3 = \infty$ and the related asymptotic laws are $\left|z_1^{(n)} - z_2^{(n)}\right|, \left|z_2^{(n)} - z_3^{(n)}\right| \sim (3/4)^n$; $\left|z_1^{(n)} + z_2^{(n)} + z_3^{(n)}\right| \sim (3/2)^n$.

In this subspace it is particularly easy to find the general structure of the backward orbit of the point ∞. We work in three dimensions, in the space (z_1, z_2, z_3), and call $\mathcal{T}$ the map (17): $\mathcal{T}(z_1, z_2, z_3) = \mathcal{M}(z_1, z_2, z_3, z_1, z_2, z_3)$. We study the set of the points $\mathcal{C} = \{(z_1, z_2, z_3) : \mathcal{T}^n(z_1, z_2, z_3) = \infty \text{ for some } n\}$; as in section 3 we define $\mathcal{C}_n = \{(z_1, z_2, z_3) : T^{n+1}(z_1, z_2, z_3) = \infty\}$, so that $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$. $\mathcal{C}$ is a subset of the set $\mathcal{O}$ defined in par. II.B.

The set $\mathcal{C}_0$ consists of the points that make the denominator of $\mathcal{T}$ vanish, that is the union of the three planes $\mathcal{C}_0^{\mathrm{I}} : z_1 + z_2 = 0, \mathcal{C}_0^{\mathrm{II}} : z_1 + z_3 = 0, \mathcal{C}_0^{\mathrm{III}} : z_2 + z_3 = 0$. We then impose $\mathcal{T}(\mathcal{C}_1) = \mathcal{C}_0$ (so that $\mathcal{T}^2(\mathcal{C}_1) = \infty$) and find that $\mathcal{C}_1$ is the union of the three surfaces $\mathcal{C}_1^{\mathrm{I}} : 2z_1z_2 + z_1z_3 + z_2z_3 = 0, \mathcal{C}_1^{\mathrm{II}} : z_1z_2 + 2z_1z_3 + z_2z_3 = 0, \mathcal{C}_1^{\mathrm{III}} : z_1z_2 + 2z_1z_3 + z_2z_3 = 0$. We proceed iteratively: at each step $\mathcal{T}\left(\mathcal{C}_{n+1}^i\right) = \mathcal{C}_n^i$ $(i = \mathrm{I, II, III})$ and $\mathcal{C}_n = \mathcal{C}_n^{\mathrm{I}} \cup \mathcal{C}_n^{\mathrm{II}} \cup \mathcal{C}_n^{\mathrm{III}}$. In general $\mathcal{C}_n^i$ is a homogeneous algebraic surface of order 2^n and has a complex structure with multiple points. Due to the homogeneity, $\mathcal{C}_n^i$ (and hence $\mathcal{C}_n$ and $\mathcal{C}$) is a generalized cone, the axis of which is the line $z_1 = z_2 = z_3$. $\mathcal{C}$ can be visualized by sectioning it with a plane orthogonal to its axis, which was done in figure 6 (here the plane is $z_1 + z_2 + z_3 = 1$).

3.3.1. An explicit solution. If we eliminate one more variable from (17) by setting $z_2 = z_3$ we get a map that can be explicitly solved for every point of the orbit. Defining $t_n = z_1^{(n+1)} \big/ z_1^{(n)}, u_n = z_2^{(n)} \big/ z_1^{(n)}$ we have

$$u_{n+1} = \frac{2u_n}{u_n + 1} \qquad t_{n+1} = \frac{2u_n + 1}{u_n + 1}. \tag{18}$$

The first is a Möbius transformation (that is, a one-dimensional rational map) the solution of which is

$$u_n = \frac{2^n u_0}{(2^n - 1)u_0 + 1} \tag{19}$$

and coming back to the original variables the result is

$$\begin{aligned} z_1^{(n)} &= \frac{2^n z_1^{(0)}}{z_2^{(0)} - z_1^{(0)} + 2^n z_1^{(0)}} z_2^{(n)} \\ z_2^{(n)} &= \prod_{k=0}^{n-1} \frac{3 \cdot 2^k z_1^{(0)} + z_2^{(0)} - z_1^{(0)}}{2 \cdot 2^k z_1^{(0)} + z_2^{(0)} - z_1^{(0)}}. \end{aligned} \tag{20}$$

As a consequence, the points whose nth iterate is infinite are those such that $z_2^{(0)} \big/ z_1^{(0)} = 1 - 2^n$.

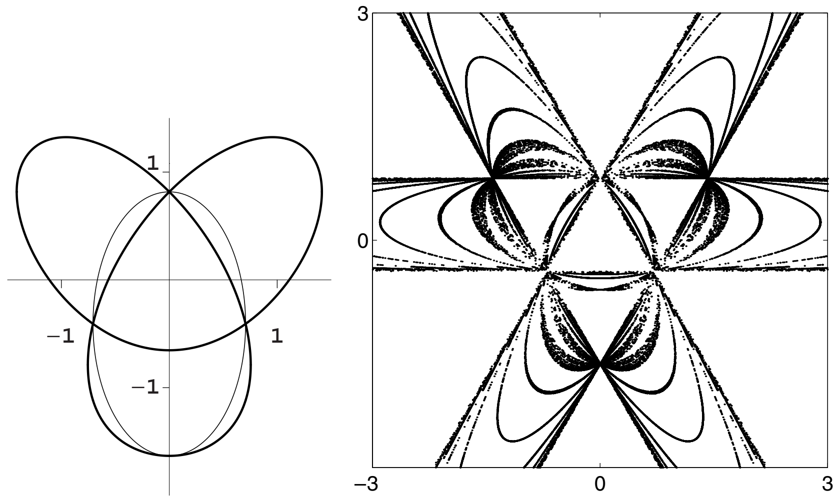

Figure 6. The cone $\mathcal{C}$ sectioned by the plane $z_1 + z_2 + z_3 = 1$. Left: the intersection of the sets $\mathcal{C}_1^{\mathrm{I}}$ (thin line) and $\mathcal{C}_2^{\mathrm{I}}$ (thick line) with the plane. Right: the (numerically computed) intersection of the whole set $\mathcal{C}$ with the plane. The triangle in the centre is the intersection of $\mathcal{C}$ with the octant $z_1, z_2, z_3 > 0$. The figure is oriented so that the horizontal and vertical axes coincide with the intersections of the plane with $z_2 = 1/3$ and $z_1 = z_3$, respectively.

3.4. Subspace **IV***:* $(z_1, z_2, z_2, z_4, z_5, z_5)$

The attracting fixed point of the map is $z_1 = z_2 = z_4 = z_5 = \infty$ and the asymptotic laws are

$$\begin{aligned} &|z_1 + 2z_2 + z_4 + 2z_5| \sim (3/2)^n \\ &|z_1 - z_4|, |z_2 - z_5| \sim (5/4)^n \\ &|z_1 - z_2 + z_4 - z_5| \sim (3/4)^n. \end{aligned} \tag{21}$$

The relations between the external impedances are $\mathcal{Z}_{bc} = \mathcal{Z}_{ac}$, $\mathcal{Z}_{ad} = \mathcal{Z}_{bd}$.

3.5. Subspace **V***:* $(z_1, z_2, -z_2, z_4, z_5, -z_5)$

In this case the submap is too long to be reported. The external impedances are $\mathcal{Z}_{ab} = 0$, $\mathcal{Z}_{ac} = -\mathcal{Z}_{bc}$ and $\mathcal{Z}_{ad} = -\mathcal{Z}_{bd}$; $\mathcal{Z}_{bc} = z_4$.

It can be shown that the quantity $\tilde{z} = 2z_2^{(n)}z_5^{(n)}/\big(z_5^{(n)} - z_2^{(n)}\big)$ is constant; therefore, as for submap (15), the orbit of the system lies on a surface determined by the initial conditions: $2z_2^{(n)}z_5^{(n)}/\big(z_5^{(n)} - z_2^{(n)}\big) = 2z_2^{(0)}z_5^{(0)}/\big(z_5^{(0)} - z_2^{(0)}\big) = \tilde{z}$. The attracting fixed point of the submap is $(z_1, z_2, z_4, z_5) = (0, \tilde{z}, \infty, -\tilde{z})$; the asymptotic laws are

$$\begin{aligned} &\big|z_1^{(n)}\big| \sim ((1/4)(7 - \sqrt{17}))^n = (0.719\,22\ldots)^n \\ &\big|z_4^{(n)}\big| \sim ((1/8)(7 + \sqrt{17}))^n = (1.390\,39\ldots)^n \\ &\big|z_2^{(n)} - \tilde{z}\big|, \big|z_5^{(n)} + \tilde{z}\big| \sim ((1/2)(6 - \sqrt{17}))^n = (0.938\,44\ldots)^n. \end{aligned} \tag{22}$$

The product $z_1 z_4$ converges to the value $z_1^{(\infty)}z_4^{(\infty)} = (1/4)(1 - \sqrt{17})\tilde{z}^2$ with the law

$$\big|z_1^{(n)}z_4^{(n)} - z_1^{(\infty)}z_4^{(\infty)}\big| \sim ((5/8)(5 - \sqrt{17}))^n = (0.548\,06\ldots)^n. \tag{23}$$

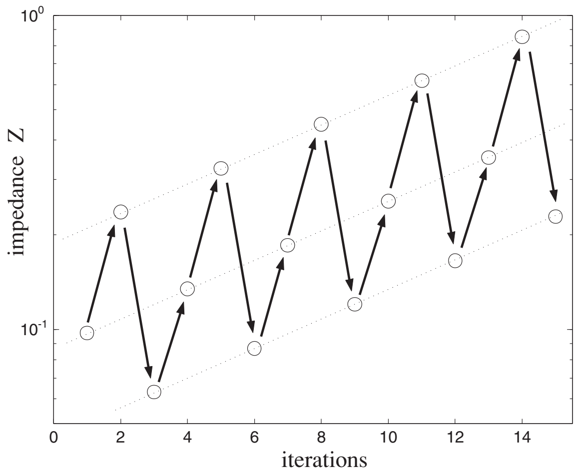

Figure 7. Oscillating asymptotic behaviour with $p = 3$.

4. Oscillating asymptotic behaviour

A phenomenon has been previously introduced [5] that concerns the periodic points of the map for the variables' ratios. Consider a two-dimensional rational map $(x_{n+1}, y_{n+1}) = \mathcal{R}(x_n, y_n)$, and define $u_n = y_n/x_n$ and $t_n = x_{n+1}/x_n$. We can always decouple the map for these new variables in such a way that $u_{n+1} = U(u_n)$ and $t_n = T(u_n)$, where U and T are rational maps; this way we can limit ourselves to study the map U. Suppose now that we find a periodic point u_0 of period p: $u_p = U^p(u_0) = u_0$; then we also get a periodic point for the map T: $t_p = T(u_0) = t_0$. Turning back to the original variables, from the periodicity of t we obtain

$$\frac{x_{p+1}}{x_p} = \frac{x_1}{x_0}; \frac{x_{p+2}}{x_{p+1}} = \frac{x_2}{x_1}; \qquad (\cdots)\frac{x_{p+k+1}}{x_{p+k}} = \frac{x_{k+1}}{x_k}; (\cdots)\frac{x_{mp+k+1}}{x_{mp+k}} = \frac{x_{k+1}}{x_k}$$

for any positive integer m and $k = 0, 1, \ldots, p-1$. Dividing term by term

$$\frac{x_p}{x_0} = \frac{x_{p+1}}{x_1} = \frac{x_{p+2}}{x_2} = (\cdots) = \frac{x_{p+k}}{x_k} = (\cdots) = \frac{x_{mp+k}}{x_{(m-1)p+k}} \tag{24}$$

so

$$x_{mp+k} = x_{(m-1)p+k}\left(\frac{x_p}{x_0}\right) = x_{(m-2)p+k}\left(\frac{x_p}{x_0}\right)^2 = (\cdots) = \left(\frac{x_p}{x_0}\right)^m x_k. \tag{25}$$

This means that the orbit of the system splits into p different branches, one for each of the first p points. All branches display the same power-law behaviour, $x_n \sim (x_p/x_0)^{\frac{n}{p}}$, but starting from a different point; at every step the system jumps from one branch to another: we call this *oscillating asymptotic behaviour*. In a logarithmic plot of x_n against n this results in p different straight lines, as exemplified in figure 7 for $p = 3$. A compact form for the overall asymptotic law is

$$x_n \sim f_p(n)(a_p)^n \tag{26}$$

where $a_p = (x_p/x_0)^{1/p}$ and $f_p(n)$ is a p-periodic function of n. An analogous asymptotic law holds for y_n.

These conclusions can be easily generalized to maps with more than two variables. In our case the map represents the impedances on the links of the fractal, and an OAB for the map coincides with an OAB for the external impedances.

It is well known [9] that a rational map of degree d can have no more than $2d-2$ attracting periodic points, while the number of its repelling periodic points is infinite. The closure of the repelling periodic points set is called the Julia set of the map; so, the properties of the OAB points of a map are in close connection with the Julia set of the map for its ratios. Since the OAB takes place in coincidence with repelling periodic points, and in physical systems we cannot start with infinite precision from a given point (i.e., a given set of impedances), this kind of behaviour persists only for a finite number of iterations, after which the system evolves towards an attracting point. So this effect is destroyed in the asymptotic limit: it is a feature of limited-size, or mesoscopic, systems.

In several models on self-similar structures [10] the log-periodic corrections to scaling, in the form of equation (29), are shown to be a signature of the dilatation-invariant geometry of the underlying system. In particular, Derrida *et al* [11] connected such corrections (and called them 'oscillatory critical amplitudes') to the Julia set of the renormalization group map for statistical models on hierarchical lattices. We point out that those phenomena are characteristic of the thermodynamic limit, as opposed to ours that can only occur far from that limit.

In the following we examine the OAB for some submaps of $\mathcal{M}$.

4.1. Subspace *I*

We define $u_n = z_1^{(n)} \big/ z_4^{(n)}$ and $t_n = z_4^{(n+1)} \big/ z_4^{(n)}$. The equation for u_n decouples and t_n evolves only through u_n:

$$\begin{aligned} u_{n+1} &= U(u_n) = \frac{7u_n^2 + 5u_n}{u_n^2 + 7u_n + 4} \\ t_n &= T(u_n) = \frac{2u_n + 4}{u_n + 3}. \end{aligned} \tag{27}$$

The attracting fixed point of U is 1, that corresponds to $z_1 = z_4 = \infty$ of map (14). There are in general $2^p + 1$ periodic points of period p; it can be shown that almost all are repelling. A general property of rational maps [5, 9] states indeed that if an attracting periodic point exists, then its basin of attraction contains at least one critical point (a point u such that $U'(u) = 0$). Since the only critical points of the map U are $(-7 \pm \mathrm{i}\sqrt{6})/11$ and they fall in the basin of attraction of the fixed point 1, there are no attracting periodic points apart from 1. Thus, all the points we find are repelling periodic (or at most periodic indifferent, but we have found none). Furthermore, the Julia set of a map can be constructed by tracing the backward orbit of just one of its elements (here we can take, e.g., the fixed point -1). So, since the preimages $U_\pm(u)$ are $(5 - 7u \pm \sqrt{33u^2 + 42u + 35})/(2u - 14)$ and map the interval $(-1, 0)$ onto itself, the Julia set of the map U is contained in that interval. In figure 7 we show the evolution of the external impedance $\mathcal{Z}_{ad}$ corresponding to the period-3 OAB starting point $z_1^{(0)} \simeq -0.904$, $z_4^{(0)} = 1$; in this case the constant a_p in equation (26) has the value $a_3 \simeq 1.379$.

4.2. Subspace *II*

We define again $u_n = z_1^{(n)} \big/ z_4^{(n)}$ and $t_n = z_4^{(n+1)} \big/ z_4^{(n)}$ and find the reduced map

$$\begin{aligned} u_{n+1} &= U(u_n) = \frac{-u_n(u_n + 3)}{u_n^2 - u_n - 4} \\ t_n &= T(u_n) = \frac{u_n + 2}{u_n + 3}. \end{aligned} \tag{28}$$

The unique attracting fixed point of U is 0 (corresponding to the point $(z_1, z_4) = (\tilde{z}, \infty)$ of map (15)). There are in general $2^p + 1$ points of period p; since the preimages of U are

$U_{\pm}(u) = (3 - u \pm \sqrt{17u^2 + 10u + 9})/(-2u - 2)$ and map the real axis onto itself, the Julia set of the map U is contained in $\mathbf{R}$. There are no attracting periodic points because the only critical points $-1 \pm i\sqrt{2}$ belong to the basin of attraction of the fixed point 0.

4.3. *Subspace* ***III***

We define $t_n = z_1^{(n+1)}/z_1^{(n)}$, $u_n = z_2^{(n)}/z_1^{(n)}$ and $w_n = z_3^{(n)}/z_1^{(n)}$; the evolution of t_n decouples, depending only on u_n, w_n, and we are led to the equations

$$\begin{aligned} u_{n+1} &= u_n \frac{1 + u_n}{u_n + w_n} \\ w_{n+1} &= w_n \frac{1 + w_n}{u_n + w_n} \\ t_{n+1} &= T(u_n, w_n) = \frac{2(u_n + w_n + u_n w_n)}{(t_n + u_n)(t_n + w_n)}. \end{aligned} \tag{29}$$

As we have said above, all the considerations of the one-dimensional case hold: if we find a periodic point for the map $(u_n, w_n) \to (u_{n+1}, w_{n+1})$ we have also found a nth-order OAB point. The map being two-dimensional, despite its simplicity, the task is now much more difficult. In general it can be said that there are *at most* 2^{p+1} periodic points of period p for the map. We have found with numerical methods several of them up to order 4, and some of them are complex.

5. Frequency dependence

The system discussed in this paper can be implemented by putting frequency-dependent impedances on the links of the gasket and measuring the external impedances as a function of the frequency at some given generation n. This could both represent the frequency-dependent response of a disordered system (in the thermodynamic limit $n \to \infty$) and a man-made circuit with passive elements (in the low-n regime). In the following two sections, we first choose subspace **I** to show what appearance such a response could have in both cases (other low-dimensional submaps show similar behaviours). Then we turn to subspace **III**, that in the thermodynamic limit has a quite different and interesting response.

We work with real numbers, keeping in mind that (modulo imaginary units) they correspond to pure inductances and capacitances. There is no loss of generality in doing so: if we replace pure impedances with realistic ones (typically with a small resistance in series or in parallel), the resonance poles on the frequency axis become resonance peaks centred roughly on the same points.

5.1. *Subspace* ***I***

We start from the configuration

$$z_1^{(0)} = -\frac{1}{\omega C}; \qquad z_4^{(0)} = \omega L \tag{30}$$

then iterate map (14) n times and measure one of the external impedances (henceforth, the response Z) of the fractal to a variable-frequency input at generation n. The response is shown in figure 8 (left) for a very big system ($n = 50$); as a consequence of map (14), at generation n there are $2^{n+1} - 2$ poles on the frequency axis. For $0 < \omega < \bar{\omega}$, with $\bar{\omega} = 1/\sqrt{LC}$, the response is smooth (without poles). In the range of frequencies from the value $\bar{\omega}$ to ∞ we find a fractal distribution of poles with box-counting dimension 0.72 ± 0.01 (independent of

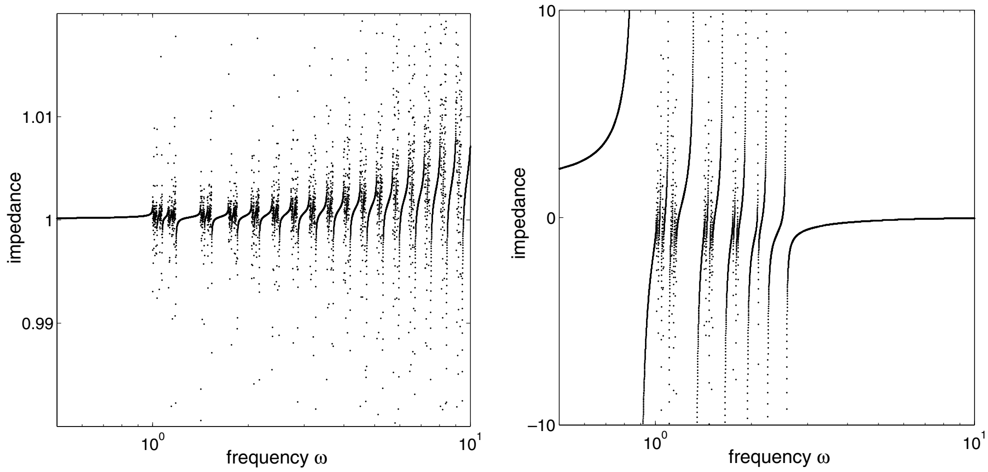

Figure 8. Dependence of the electrical response on the frequency for a 3d gasket belonging to subspace **I**. Left: in the case of a very big system ($n = 50$). Right: in the case of a small system ($n = 5$).

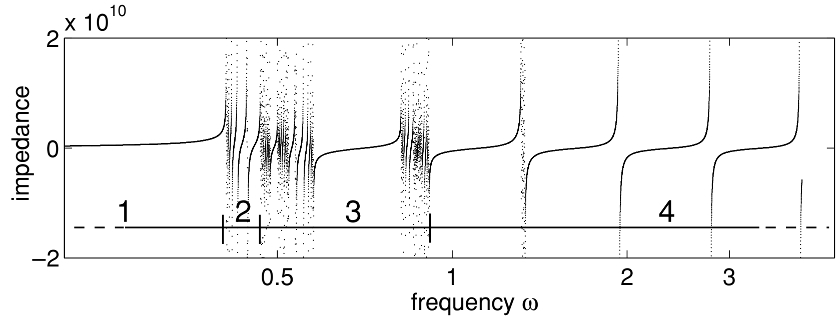

Figure 9. Dependence of the electrical response on the frequency for a 3d gasket belonging to subspace **III** for a very big system ($n = 50$). The behaviour in each of the four regions is explained in the text.

L and C). For small systems ($n \leqslant 5$, so with a number of elements $\lesssim 6 \times 10^3$; figure 8, right) the region with poles does not extend to ∞ and the response becomes smooth for $\omega \to \infty$ with an inductive law ($Z \sim \omega$).

If we start instead with $z_1^{(0)} = \omega L, z_4^{(0)} = -1/\omega C$, the response displays a fractal distribution of poles (with box-counting dimension 0.685 ± 0.01) for $0 < \omega < \bar{\omega}$ and is smooth for $\omega > \bar{\omega}$. For systems with small n the non-smooth region does not reach 0, where we have rather a capacitive law: $Z \sim -1/\omega$.

5.2. *Subspace* ***III***

We choose $z_1^{(0)} = \omega L_1, z_2^{(0)} = \omega L_2, z_3^{(0)} = 1/\omega C$, iterate equations (17) and measure the final impedance. For a large system with $n = 50$, as shown in figure 9, the response as a function of the frequency is quite complicated. We distinguish four regions, which we enumerate and describe from left to right. In region 1, that goes from 0 to $\tilde{\omega} = 1/\sqrt{\tilde{L}C}$ with $\tilde{L} = \max(L_1, L_2)$, the response is smooth. On the right of $\tilde{\omega}$ we find region 2, that shows small fractal sets of poles localized around 'main' poles. If we label with ω_p the positions of the main poles from right to left we find that their distribution satisfies two alternating geometric progressions: for 'odd' poles $\omega_{2p+1} - \tilde{\omega} \sim 2^{-p}$, and for 'even' poles $\omega_{2p} - \tilde{\omega} \sim 2^{-p}$, with a fixed ratio $\omega_{2p}/\omega_{2p+1} \simeq 1.1$. In region 3 the distribution of the poles is fractal with a box-counting dimension $d_F = 0,485 \pm 0.01$ (independent of $L_{1,2}$ and C). In region 4, that extends up to ∞, the poles show again a double geometric progression; labelling the positions of the poles with ω_p from left to right we find $\omega_{2p+1} \sim 2^{p/2}$, $\omega_{2p} \sim 2^{p/2}$. For smaller systems the main features of the above-described regions remain, but region 4 does not reach ∞, where we have instead an inductive behaviour ($Z \sim \omega$).

In the second case, that is starting from $z_1^{(0)} = 1/\omega C_1$, $z_2^{(0)} = 1/\omega C_2$, $z_3^{(0)} = \omega L$, the behaviour is similar; we describe it from left to right without a figure. In region 1, starting from 0, we find a double geometric progression of poles; labelling the poles from right to left we have $\omega_{2p+1} \sim 2^{-p/2}$, $\omega_{2p} \sim 2^{-p/2}$ (so 0 is an accumulation point of poles). In region 2 we find a fractal distribution of poles with the same d_F as in region 3 of the first case. In region 3, that ends in $\tilde{\omega} = 1/\sqrt{L\tilde{C}}$, with $\tilde{C} = \min(C_1, C_2)$, there is again a double geometric progression; labelling the poles from left to right we have $\omega_{2p+1} - \tilde{\omega} \sim 2^{-p}$, $\omega_{2p} - \tilde{\omega} \sim 2^{-p}$. Region 4 is smooth with a capacitive behaviour: $Z \sim -1/\omega$. For small size region 1 does not approximate 0; rather, for $\omega \to 0$ the law is inductive: $Z \sim \omega$.

6. Conclusions and perspectives

We addressed the issue of the decimation of a 3d Sierpinski gasket of impedances and found the exact decimation map. This result allowed us to describe several features that make this system more complicated and interesting than its two-dimensional analogue. Among these there are a set of submaps with conserved quantities, and another set where an explicit solution can be found for every point of the orbit.

This result is relevant not only to mathematics: indeed, it enables us to study properties that can be measured in real circuits. In particular, we were able to find the distribution of the resonances of the system in the impedance space, the dependence of the external impedance on the frequency of an applied signal, and typical small-size effects like oscillating asymptotic behaviour.

The method we used in this paper can be readily generalized to the n-dimensional version of the Sierpinski gasket, where the basic cell is a hypertetrahedron. The exact calculations become rapidly unaffordable, as can be guessed from the difference between the solutions in the 2d and the 3d cases. However, some general results can be drawn and to first order a general solution for the asymptotic regime can be found, as we will show in a forthcoming work [12].

Appendix A.

$$
\begin{aligned}
P_{12}(z_1, z_2, z_3, z_4, z_5, z_6) = {} & 6\, z_1^3 z_2^3 z_3^3 z_4^2 z_5 + 6\, z_1^3 z_2^3 z_3^2 z_4^3 z_5 + 6\, z_1^3 z_2^2 z_3^3 z_4^3 z_5 + 6\, z_1^3 z_2^3 z_3^3 z_4 z_5^2 \\
& + 16\, z_1^3 z_2^3 z_3^2 z_4^2 z_5^2 + 12\, z_1^3 z_2^2 z_3^3 z_4^2 z_5^2 + 12\, z_1^2 z_2^3 z_3^3 z_4^2 z_5^2 + 8\, z_1^3 z_2^3 z_3 z_4^3 z_5^2 \\
& + 16\, z_1^3 z_2^2 z_3^2 z_4^3 z_5^2 + 12\, z_1^2 z_2^3 z_3^2 z_4^3 z_5^2 + 6\, z_1^3 z_2 z_3^3 z_4^3 z_5^2 + 12\, z_1^2 z_2^2 z_3^3 z_4^3 z_5^2 \\
& + 6\, z_1^3 z_2^3 z_3^2 z_4 z_5^3 + 6\, z_1^2 z_2^3 z_3^3 z_4 z_5^3 + 8\, z_1^3 z_2^3 z_3 z_4^2 z_5^3 + 12\, z_1^3 z_2^2 z_3^2 z_4^2 z_5^3 \\
& + 16\, z_1^2 z_2^3 z_3^2 z_4^2 z_5^3 + 12\, z_1^2 z_2^2 z_3^3 z_4^2 z_5^3 + 6\, z_1 z_2^3 z_3^3 z_4^2 z_5^3 + 8\, z_1^3 z_2^2 z_3 z_4^3 z_5^3 \\
& + 8\, z_1^2 z_2^3 z_3 z_4^3 z_5^3 + 6\, z_1^3 z_2 z_3^2 z_4^3 z_5^3 + 16\, z_1^2 z_2^2 z_3^2 z_4^3 z_5^3 + 6\, z_1 z_2^3 z_3^2 z_4^3 z_5^3 \\
& + 6\, z_1^2 z_2 z_3^3 z_4^3 z_5^3 + 6\, z_1 z_2^2 z_3^3 z_4^3 z_5^3 + 6\, z_1^3 z_2^3 z_3^3 z_4^2 z_6 + 6\, z_1^3 z_2^3 z_3^2 z_4^3 z_6 \\
& + 6\, z_1^3 z_2^2 z_3^3 z_4^3 z_6 + 13\, z_1^3 z_2^3 z_3^3 z_4 z_5 z_6 + 32\, z_1^3 z_2^3 z_3^2 z_4^2 z_5 z_6 + 32\, z_1^3 z_2^2 z_3^3 z_4^2 z_5 z_6
\end{aligned}
$$

$$
\begin{aligned}
&+26\,z_1^2 z_2^3 z_3^3 z_4^2 z_5 z_6 + 16\,z_1^3 z_2^3 z_3 z_4^3 z_5 z_6 + 36\,z_1^3 z_2^2 z_3^2 z_4^3 z_5 z_6 \\
&+26\,z_1^2 z_2^3 z_3^2 z_4^3 z_5 z_6 + 16\,z_1^3 z_2 z_3^3 z_4^3 z_5 z_6 + 26\,z_1^2 z_2^2 z_3^3 z_4^3 z_5 z_6 \\
&+6\,z_1^3 z_2^3 z_3^3 z_5^2 z_6 + 32\,z_1^3 z_2^3 z_3^2 z_4 z_5^2 z_6 + 26\,z_1^3 z_2^2 z_3^3 z_4 z_5^2 z_6 + 32\,z_1^2 z_2^3 z_3^3 z_4 z_5^2 z_6 \\
&+36\,z_1^3 z_2^3 z_3 z_4^2 z_5^2 z_6 + 76\,z_1^3 z_2^2 z_3^2 z_4^2 z_5^2 z_6 + 76\,z_1^2 z_2^3 z_3^2 z_4^2 z_5^2 z_6 \\
&+26\,z_1^3 z_2 z_3^3 z_4^2 z_5^2 z_6 + 70\,z_1^2 z_2^2 z_3^3 z_4^2 z_5^2 z_6 + 26\,z_1 z_2^3 z_3^3 z_4^2 z_5^2 z_6 + 8\,z_1^3 z_2^3 z_4^3 z_5^2 z_6 \\
&+40\,z_1^3 z_2^2 z_3 z_4^3 z_5^2 z_6 + 36\,z_1^2 z_2^3 z_3 z_4^3 z_5^2 z_6 + 36\,z_1^3 z_2 z_3^2 z_4^3 z_5^2 z_6 \\
&+76\,z_1^2 z_2^2 z_3^2 z_4^3 z_5^2 z_6 + 26\,z_1 z_2^3 z_3^2 z_4^3 z_5^2 z_6 + 6\,z_1^3 z_3^3 z_4^3 z_5^2 z_6 + 32\,z_1^2 z_2 z_3^3 z_4^3 z_5^2 z_6 \\
&+26\,z_1 z_2^2 z_3^3 z_4^3 z_5^2 z_6 + 6\,z_1^3 z_2^3 z_3^2 z_5^3 z_6 + 6\,z_1^2 z_2^3 z_3^3 z_5^3 z_6 \\
&+16\,z_1^3 z_2^3 z_3 z_4 z_5^3 z_6 + 26\,z_1^3 z_2^2 z_3^2 z_4 z_5^3 z_6 + 36\,z_1^2 z_2^3 z_3^2 z_4 z_5^3 z_6 \\
&+26\,z_1^2 z_2^2 z_3^3 z_4 z_5^3 z_6 + 16\,z_1 z_2^3 z_3^3 z_4 z_5^3 z_6 + 8\,z_1^3 z_2^3 z_4^2 z_5^3 z_6 \\
&+36\,z_1^3 z_2^2 z_3 z_4^2 z_5^3 z_6 + 40\,z_1^2 z_2^3 z_3 z_4^2 z_5^3 z_6 + 26\,z_1^3 z_2 z_3^2 z_4^2 z_5^3 z_6 \\
&+76\,z_1^2 z_2^2 z_3^2 z_4^2 z_5^3 z_6 + 36\,z_1 z_2^3 z_3^2 z_4^2 z_5^3 z_6 + 26\,z_1^2 z_2 z_3^3 z_4^2 z_5^3 z_6 \\
&+32\,z_1 z_2^2 z_3^3 z_4^2 z_5^3 z_6 + 6\,z_2^3 z_3^3 z_4^2 z_5^3 z_6 + 8\,z_1^3 z_2^2 z_4^3 z_5^3 z_6 \\
&+8\,z_1^2 z_2^3 z_4^3 z_5^3 z_6 + 16\,z_1^3 z_2 z_3 z_4^3 z_5^3 z_6 + 36\,z_1^2 z_2^2 z_3 z_4^3 z_5^3 z_6 \\
&+16\,z_1 z_2^3 z_3 z_4^3 z_5^3 z_6 + 6\,z_1^3 z_3^2 z_4^3 z_5^3 z_6 + 32\,z_1^2 z_2 z_3^2 z_4^3 z_5^3 z_6 \\
&+32\,z_1 z_2^2 z_3^2 z_4^3 z_5^3 z_6 + 6\,z_2^3 z_3^2 z_4^3 z_5^3 z_6 + 6\,z_1^2 z_3^3 z_4^3 z_5^3 z_6 \\
&+13\,z_1 z_2 z_3^3 z_4^3 z_5^3 z_6 + 6\,z_2^2 z_3^3 z_4^3 z_5^3 z_6 + 6\,z_1^3 z_2^3 z_3^3 z_4 z_6^2 \\
&+12\,z_1^3 z_2^3 z_3^2 z_4^2 z_6^2 + 16\,z_1^3 z_2^2 z_3^3 z_4^2 z_6^2 + 12\,z_1^2 z_2^3 z_3^3 z_4^2 z_6^2 \\
&+6\,z_1^3 z_2^3 z_3 z_4^3 z_6^2 + 16\,z_1^3 z_2^2 z_3^2 z_4^3 z_6^2 + 12\,z_1^2 z_2^3 z_3^2 z_4^3 z_6^2 \\
&+8\,z_1^3 z_2 z_3^3 z_4^3 z_6^2 + 12\,z_1^2 z_2^2 z_3^3 z_4^3 z_6^2 + 6\,z_1^3 z_2^3 z_3^3 z_5 z_6^2 \\
&+26\,z_1^3 z_2^3 z_3^2 z_4 z_5 z_6^2 + 32\,z_1^3 z_2^2 z_3^3 z_4 z_5 z_6^2 + 32\,z_1^2 z_2^3 z_3^3 z_4 z_5 z_6^2 \\
&+26\,z_1^3 z_2^3 z_3 z_4^2 z_5 z_6^2 + 76\,z_1^3 z_2^2 z_3^2 z_4^2 z_5 z_6^2 + 70\,z_1^2 z_2^3 z_3^2 z_4^2 z_5 z_6^2 \\
&+36\,z_1^3 z_2 z_3^3 z_4^2 z_5 z_6^2 + 76\,z_1^2 z_2^2 z_3^3 z_4^2 z_5 z_6^2 + 26\,z_1 z_2^3 z_3^3 z_4^2 z_5 z_6^2 \\
&+6\,z_1^3 z_2^3 z_4^3 z_5 z_6^2 + 36\,z_1^3 z_2^2 z_3 z_4^3 z_5 z_6^2 + 32\,z_1^2 z_2^3 z_3 z_4^3 z_5 z_6^2 \\
&+40\,z_1^3 z_2 z_3^2 z_4^3 z_5 z_6^2 + 76\,z_1^2 z_2^2 z_3^2 z_4^3 z_5 z_6^2 + 26\,z_1 z_2^3 z_3^2 z_4^3 z_5 z_6^2 \\
&+8\,z_1^3 z_3^3 z_4^3 z_5 z_6^2 + 36\,z_1^2 z_2 z_3^3 z_4^3 z_5 z_6^2 + 26\,z_1 z_2^2 z_3^3 z_4^3 z_5 z_6^2 \\
&+12\,z_1^3 z_2^3 z_3^2 z_5^2 z_6^2 + 12\,z_1^3 z_2^2 z_3^3 z_5^2 z_6^2 + 16\,z_1^2 z_2^3 z_3^3 z_5^2 z_6^2 \\
&+26\,z_1^3 z_2^3 z_3 z_4 z_5^2 z_6^2 + 70\,z_1^3 z_2^2 z_3^2 z_4 z_5^2 z_6^2 + 76\,z_1^2 z_2^3 z_3^2 z_4 z_5^2 z_6^2 \\
&+26\,z_1^3 z_2 z_3^3 z_4 z_5^2 z_6^2 + 76\,z_1^2 z_2^2 z_3^3 z_4 z_5^2 z_6^2 + 36\,z_1 z_2^3 z_3^3 z_4 z_5^2 z_6^2 \\
&+12\,z_1^3 z_2^3 z_4^2 z_5^2 z_6^2 + 76\,z_1^3 z_2^2 z_3 z_4^2 z_5^2 z_6^2 + 76\,z_1^2 z_2^3 z_3 z_4^2 z_5^2 z_6^2 \\
&+76\,z_1^3 z_2 z_3^2 z_4^2 z_5^2 z_6^2 + 176\,z_1^2 z_2^2 z_3^2 z_4^2 z_5^2 z_6^2 + 76\,z_1 z_2^3 z_3^2 z_4^2 z_5^2 z_6^2 \\
&+12\,z_1^3 z_3^3 z_4^2 z_5^2 z_6^2 + 76\,z_1^2 z_2 z_3^3 z_4^2 z_5^2 z_6^2 + 76\,z_1 z_2^2 z_3^3 z_4^2 z_5^2 z_6^2 \\
&+12\,z_2^3 z_3^3 z_4^2 z_5^2 z_6^2 + 16\,z_1^3 z_2^2 z_4^3 z_5^2 z_6^2 + 16\,z_1^2 z_2^3 z_4^3 z_5^2 z_6^2 \\
&+36\,z_1^3 z_2 z_3 z_4^3 z_5^2 z_6^2 + 76\,z_1^2 z_2^2 z_3 z_4^3 z_5^2 z_6^2 + 32\,z_1 z_2^3 z_3 z_4^3 z_5^2 z_6^2 \\
&+16\,z_1^3 z_3^2 z_4^3 z_5^2 z_6^2 + 76\,z_1^2 z_2 z_3^2 z_4^3 z_5^2 z_6^2 + 70\,z_1 z_2^2 z_3^2 z_4^3 z_5^2 z_6^2 \\
&+12\,z_2^3 z_3^2 z_4^3 z_5^2 z_6^2 + 16\,z_1^2 z_3^3 z_4^3 z_5^2 z_6^2 + 32\,z_1 z_2 z_3^3 z_4^3 z_5^2 z_6^2 \\
&+12\,z_2^2 z_3^3 z_4^3 z_5^2 z_6^2 + 6\,z_1^3 z_2^3 z_3 z_5^3 z_6^2 + 12\,z_1^3 z_2^2 z_3^2 z_5^3 z_6^2 \\
&+16\,z_1^2 z_2^3 z_3^2 z_5^3 z_6^2 + 12\,z_1^2 z_2^2 z_3^3 z_5^3 z_6^2 + 8\,z_1 z_2^3 z_3^3 z_5^3 z_6^2 \\
&+6\,z_1^3 z_2^3 z_4 z_5^3 z_6^2 + 32\,z_1^3 z_2^2 z_3 z_4 z_5^3 z_6^2 + 36\,z_1^2 z_2^3 z_3 z_4 z_5^3 z_6^2
\end{aligned}
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\begin{aligned}
&+26\, z_1^3\, z_2\, z_3^2\, z_4\, z_5^3\, z_6^2 + 76\, z_1^2\, z_2^2\, z_3^2\, z_4\, z_5^3\, z_6^2 + 40\, z_1\, z_2^3\, z_3^2\, z_4\, z_5^3\, z_6^2 \\
&+26\, z_1^2\, z_2\, z_3^3\, z_4\, z_5^3\, z_6^2 + 36\, z_1\, z_2^2\, z_3^3\, z_4\, z_5^3\, z_6^2 + 8\, z_2^3\, z_3^3\, z_4\, z_5^3\, z_6^2 \\
&+16\, z_1^3\, z_2^2\, z_4^2\, z_5^3\, z_6^2 + 16\, z_1^2\, z_2^3\, z_4^2\, z_5^3\, z_6^2 + 32\, z_1^3\, z_2\, z_3\, z_4^2\, z_5^3\, z_6^2 \\
&+76\, z_1^2\, z_2^2\, z_3\, z_4^2\, z_5^3\, z_6^2 + 36\, z_1\, z_2^3\, z_3\, z_4^2\, z_5^3\, z_6^2 + 12\, z_1^3\, z_3^2\, z_4^2\, z_5^3\, z_6^2 \\
&+70\, z_1^2\, z_2\, z_3^2\, z_4^2\, z_5^3\, z_6^2 + 76\, z_1\, z_2^2\, z_3^2\, z_4^2\, z_5^3\, z_6^2 + 16\, z_2^3\, z_3^2\, z_4^2\, z_5^3\, z_6^2 \\
&+12\, z_1^2\, z_3^3\, z_4^2\, z_5^3\, z_6^2 + 32\, z_1\, z_2\, z_3^3\, z_4^2\, z_5^3\, z_6^2 + 16\, z_2^2\, z_3^3\, z_4^2\, z_5^3\, z_6^2 \\
&+6\, z_1^3\, z_2\, z_4^3\, z_5^3\, z_6^2 + 12\, z_1^2\, z_2^2\, z_4^3\, z_5^3\, z_6^2 + 6\, z_1\, z_2^3\, z_4^3\, z_5^3\, z_6^2 \\
&+6\, z_1^3\, z_3\, z_4^3\, z_5^3\, z_6^2 + 26\, z_1^2\, z_2\, z_3\, z_4^3\, z_5^3\, z_6^2 + 26\, z_1\, z_2^2\, z_3\, z_4^3\, z_5^3\, z_6^2 \\
&+6\, z_2^3\, z_3\, z_4^3\, z_5^3\, z_6^2 + 12\, z_1^2\, z_3^2\, z_4^3\, z_5^3\, z_6^2 + 26\, z_1\, z_2\, z_3^2\, z_4^3\, z_5^3\, z_6^2 \\
&+12\, z_2^2\, z_3^2\, z_4^3\, z_5^3\, z_6^2 + 6\, z_1\, z_3^3\, z_4^3\, z_5^3\, z_6^2 + 6\, z_2\, z_3^3\, z_4^3\, z_5^3\, z_6^2 \\
&+6\, z_1^3\, z_2^2\, z_3^3\, z_4\, z_6^3 + 6\, z_1^2\, z_2^3\, z_3^3\, z_4\, z_6^3 + 12\, z_1^3\, z_2^2\, z_3^2\, z_4^2\, z_6^3 \\
&+12\, z_1^2\, z_2^3\, z_3^2\, z_4^2\, z_6^3 + 8\, z_1^3\, z_2\, z_3^3\, z_4^2\, z_6^3 + 16\, z_1^2\, z_2^2\, z_3^3\, z_4^2\, z_6^3 \\
&+6\, z_1\, z_2^3\, z_3^3\, z_4^2\, z_6^3 + 6\, z_1^3\, z_2^2\, z_3\, z_4^3\, z_6^3 + 6\, z_1^2\, z_2^3\, z_3\, z_4^3\, z_6^3 \\
&+8\, z_1^3\, z_2\, z_3^2\, z_4^3\, z_6^3 + 16\, z_1^2\, z_2^2\, z_3^2\, z_4^3\, z_6^3 + 6\, z_1\, z_2^3\, z_3^2\, z_4^3\, z_6^3 \\
&+8\, z_1^2\, z_2\, z_3^3\, z_4^3\, z_6^3 + 6\, z_1\, z_2^2\, z_3^3\, z_4^3\, z_6^3 + 6\, z_1^3\, z_2^2\, z_3^3\, z_5\, z_6^3 \\
&+6\, z_1^2\, z_2^3\, z_3^3\, z_5\, z_6^3 + 26\, z_1^3\, z_2^2\, z_3^2\, z_4\, z_5\, z_6^3 + 26\, z_1^2\, z_2^3\, z_3^2\, z_4\, z_5\, z_6^3 \\
&+16\, z_1^3\, z_2\, z_3^3\, z_4\, z_5\, z_6^3 + 36\, z_1^2\, z_2^2\, z_3^3\, z_4\, z_5\, z_6^3 + 16\, z_1\, z_2^3\, z_3^3\, z_4\, z_5\, z_6^3 \\
&+26\, z_1^3\, z_2^2\, z_3\, z_4^2\, z_5\, z_6^3 + 26\, z_1^2\, z_2^3\, z_3\, z_4^2\, z_5\, z_6^3 + 36\, z_1^3\, z_2\, z_3^2\, z_4^2\, z_5\, z_6^3 \\
&+76\, z_1^2\, z_2^2\, z_3^2\, z_4^2\, z_5\, z_6^3 + 32\, z_1\, z_2^3\, z_3^2\, z_4^2\, z_5\, z_6^3 + 8\, z_1^3\, z_3^3\, z_4^2\, z_5\, z_6^3 \\
&+40\, z_1^2\, z_2\, z_3^3\, z_4^2\, z_5\, z_6^3 + 36\, z_1\, z_2^2\, z_3^3\, z_4^2\, z_5\, z_6^3 + 6\, z_2^3\, z_3^3\, z_4^2\, z_5\, z_6^3 \\
&+6\, z_1^3\, z_2^2\, z_4^3\, z_5\, z_6^3 + 6\, z_1^2\, z_2^3\, z_4^3\, z_5\, z_6^3 + 16\, z_1^3\, z_2\, z_3\, z_4^3\, z_5\, z_6^3 \\
&+32\, z_1^2\, z_2^2\, z_3\, z_4^3\, z_5\, z_6^3 + 13\, z_1\, z_2^3\, z_3\, z_4^3\, z_5\, z_6^3 + 8\, z_1^3\, z_3^2\, z_4^3\, z_5\, z_6^3 \\
&+36\, z_1^2\, z_2\, z_3^2\, z_4^3\, z_5\, z_6^3 + 32\, z_1\, z_2^2\, z_3^2\, z_4^3\, z_5\, z_6^3 + 6\, z_2^3\, z_3^2\, z_4^3\, z_5\, z_6^3 \\
&+8\, z_1^2\, z_3^3\, z_4^3\, z_5\, z_6^3 + 16\, z_1\, z_2\, z_3^3\, z_4^3\, z_5\, z_6^3 + 6\, z_2^2\, z_3^3\, z_4^3\, z_5\, z_6^3 \\
&+12\, z_1^3\, z_2^2\, z_3^2\, z_5^2\, z_6^3 + 12\, z_1^2\, z_2^3\, z_3^2\, z_5^2\, z_6^3 + 6\, z_1^3\, z_2\, z_3^3\, z_5^2\, z_6^3 \\
&+16\, z_1^2\, z_2^2\, z_3^3\, z_5^2\, z_6^3 + 8\, z_1\, z_2^3\, z_3^3\, z_5^2\, z_6^3 + 26\, z_1^3\, z_2^2\, z_3\, z_4\, z_5^2\, z_6^3 \\
&+26\, z_1^2\, z_2^3\, z_3\, z_4\, z_5^2\, z_6^3 + 32\, z_1^3\, z_2\, z_3^2\, z_4\, z_5^2\, z_6^3 + 76\, z_1^2\, z_2^2\, z_3^2\, z_4\, z_5^2\, z_6^3 \\
&+36\, z_1\, z_2^3\, z_3^2\, z_4\, z_5^2\, z_6^3 + 6\, z_1^3\, z_3^3\, z_4\, z_5^2\, z_6^3 + 36\, z_1^2\, z_2\, z_3^3\, z_4\, z_5^2\, z_6^3 \\
&+40\, z_1\, z_2^2\, z_3^3\, z_4\, z_5^2\, z_6^3 + 8\, z_2^3\, z_3^3\, z_4\, z_5^2\, z_6^3 + 12\, z_1^3\, z_2^2\, z_4^2\, z_5^2\, z_6^3 \\
&+12\, z_1^2\, z_2^3\, z_4^2\, z_5^2\, z_6^3 + 32\, z_1^3\, z_2\, z_3\, z_4^2\, z_5^2\, z_6^3 + 70\, z_1^2\, z_2^2\, z_3\, z_4^2\, z_5^2\, z_6^3 \\
&+32\, z_1\, z_2^3\, z_3\, z_4^2\, z_5^2\, z_6^3 + 16\, z_1^3\, z_3^2\, z_4^2\, z_5^2\, z_6^3 + 76\, z_1^2\, z_2\, z_3^2\, z_4^2\, z_5^2\, z_6^3 \\
&+76\, z_1\, z_2^2\, z_3^2\, z_4^2\, z_5^2\, z_6^3 + 16\, z_2^3\, z_3^2\, z_4^2\, z_5^2\, z_6^3 + 16\, z_1^2\, z_3^3\, z_4^2\, z_5^2\, z_6^3 \\
&+36\, z_1\, z_2\, z_3^3\, z_4^2\, z_5^2\, z_6^3 + 16\, z_2^2\, z_3^3\, z_4^2\, z_5^2\, z_6^3 + 6\, z_1^3\, z_2\, z_4^3\, z_5^2\, z_6^3 \\
&+12\, z_1^2\, z_2^2\, z_4^3\, z_5^2\, z_6^3 + 6\, z_1\, z_2^3\, z_4^3\, z_5^2\, z_6^3 + 6\, z_1^3\, z_3\, z_4^3\, z_5^2\, z_6^3 \\
&+26\, z_1^2\, z_2\, z_3\, z_4^3\, z_5^2\, z_6^3 + 26\, z_1\, z_2^2\, z_3\, z_4^3\, z_5^2\, z_6^3 + 6\, z_2^3\, z_3\, z_4^3\, z_5^2\, z_6^3 \\
&+12\, z_1^2\, z_3^2\, z_4^3\, z_5^2\, z_6^3 + 26\, z_1\, z_2\, z_3^2\, z_4^3\, z_5^2\, z_6^3 + 12\, z_2^2\, z_3^2\, z_4^3\, z_5^2\, z_6^3 \\
&+6\, z_1\, z_3^3\, z_4^3\, z_5^2\, z_6^3 + 6\, z_2\, z_3^3\, z_4^3\, z_5^2\, z_6^3 + 6\, z_1^3\, z_2^2\, z_3\, z_5^3\, z_6^3 \\
&+6\, z_1^2\, z_2^3\, z_3\, z_5^3\, z_6^3 + 6\, z_1^3\, z_2\, z_3^2\, z_5^3\, z_6^3 + 16\, z_1^2\, z_2^2\, z_3^2\, z_5^3\, z_6^3 \\
&+8\, z_1\, z_2^3\, z_3^2\, z_5^3\, z_6^3 + 6\, z_1^2\, z_2\, z_3^3\, z_5^3\, z_6^3 + 8\, z_1\, z_2^2\, z_3^3\, z_5^3\, z_6^3
\end{aligned}
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\begin{aligned}
&+6\,z_1^3\,z_2^2\,z_4\,z_5^3\,z_6^3+6\,z_1^2\,z_2^3\,z_4\,z_5^3\,z_6^3+13\,z_1^3\,z_2\,z_3\,z_4\,z_5^3\,z_6^3\\
&+32\,z_1^2\,z_2^2\,z_3\,z_4\,z_5^3\,z_6^3+16\,z_1\,z_2^3\,z_3\,z_4\,z_5^3\,z_6^3+6\,z_1^3\,z_3^2\,z_4\,z_5^3\,z_6^3\\
&+32\,z_1^2\,z_2\,z_3^2\,z_4\,z_5^3\,z_6^3+36\,z_1\,z_2^2\,z_3^2\,z_4\,z_5^3\,z_6^3+8\,z_2^3\,z_3^2\,z_4\,z_5^3\,z_6^3\\
&+6\,z_1^2\,z_3^3\,z_4\,z_5^3\,z_6^3+16\,z_1\,z_2\,z_3^3\,z_4\,z_5^3\,z_6^3+8\,z_2^2\,z_3^3\,z_4\,z_5^3\,z_6^3\\
&+6\,z_1^3\,z_2\,z_4^2\,z_5^3\,z_6^3+12\,z_1^2\,z_2^2\,z_4^2\,z_5^3\,z_6^3+6\,z_1\,z_2^3\,z_4^2\,z_5^3\,z_6^3\\
&+6\,z_1^3\,z_3\,z_4^2\,z_5^3\,z_6^3+26\,z_1^2\,z_2\,z_3\,z_4^2\,z_5^3\,z_6^3+26\,z_1\,z_2^2\,z_3\,z_4^2\,z_5^3\,z_6^3\\
&+6\,z_2^3\,z_3\,z_4^2\,z_5^3\,z_6^3+12\,z_1^2\,z_3^2\,z_4^2\,z_5^3\,z_6^3+26\,z_1\,z_2\,z_3^2\,z_4^2\,z_5^3\,z_6^3\\
&+12\,z_2^2\,z_3^2\,z_4^2\,z_5^3\,z_6^3+6\,z_1\,z_3^3\,z_4^2\,z_5^3\,z_6^3+6\,z_2\,z_3^3\,z_4^2\,z_5^3\,z_6^3.
\end{aligned}
$$

$$
\begin{aligned}
&Q_{12}(z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6)\\
&\quad=2\,z_1^4\,z_2^3\,z_3^3\,z_4^2+2\,z_1^4\,z_2^3\,z_3^2\,z_4^3+2\,z_1^4\,z_2^2\,z_3^3\,z_4^3+2\,z_1^4\,z_2^3\,z_3^3\,z_4\,z_5+7\,z_1^4\,z_2^3\,z_3^2\,z_4^2\,z_5\\
&\qquad+5\,z_1^4\,z_2^2\,z_3^3\,z_4^2\,z_5+10\,z_1^3\,z_2^3\,z_3^3\,z_4^2\,z_5+3\,z_1^4\,z_2^3\,z_3\,z_4^3\,z_5+12\,z_1^4\,z_2^2\,z_3^2\,z_4^3\,z_5\\
&\qquad+10\,z_1^3\,z_2^3\,z_3^2\,z_4^3\,z_5+3\,z_1^4\,z_2\,z_3^3\,z_4^3\,z_5+10\,z_1^3\,z_2^2\,z_3^3\,z_4^3\,z_5+3\,z_1^4\,z_2^3\,z_3^2\,z_4\,z_5^2\\
&\qquad+7\,z_1^3\,z_2^3\,z_3^3\,z_4\,z_5^2+4\,z_1^4\,z_2^3\,z_3\,z_4^2\,z_5^2+9\,z_1^4\,z_2^2\,z_3^2\,z_4^2\,z_5^2+23\,z_1^3\,z_2^3\,z_3^2\,z_4^2\,z_5^2\\
&\qquad+16\,z_1^3\,z_2^2\,z_3^3\,z_4^2\,z_5^2+14\,z_1^2\,z_2^3\,z_3^3\,z_4^2\,z_5^2+10\,z_1^4\,z_2^2\,z_3\,z_4^3\,z_5^2+12\,z_1^3\,z_2^3\,z_3\,z_4^3\,z_5^2\\
&\qquad+6\,z_1^4\,z_2\,z_3^2\,z_4^3\,z_5^2+33\,z_1^3\,z_2^2\,z_3^2\,z_4^3\,z_5^2+14\,z_1^2\,z_2^3\,z_3^2\,z_4^3\,z_5^2+9\,z_1^3\,z_2\,z_3^3\,z_4^3\,z_5^2\\
&\qquad+14\,z_1^2\,z_2^2\,z_3^3\,z_4^3\,z_5^2+z_1^4\,z_2^3\,z_3\,z_4\,z_5^3+6\,z_1^3\,z_2^3\,z_3^2\,z_4\,z_5^3+5\,z_1^2\,z_2^3\,z_3^3\,z_4\,z_5^3\\
&\qquad+4\,z_1^4\,z_2^2\,z_3\,z_4^2\,z_5^3+6\,z_1^3\,z_2^3\,z_3\,z_4^2\,z_5^3+15\,z_1^3\,z_2^2\,z_3^2\,z_4^2\,z_5^3+16\,z_1^2\,z_2^3\,z_3^2\,z_4^2\,z_5^3\\
&\qquad+11\,z_1^2\,z_2^2\,z_3^3\,z_4^2\,z_5^3+6\,z_1\,z_2^3\,z_3^3\,z_4^2\,z_5^3+3\,z_1^4\,z_2\,z_3\,z_4^3\,z_5^3+12\,z_1^3\,z_2^2\,z_3\,z_4^3\,z_5^3\\
&\qquad+9\,z_1^2\,z_2^3\,z_3\,z_4^3\,z_5^3+9\,z_1^3\,z_2\,z_3^2\,z_4^3\,z_5^3+21\,z_1^2\,z_2^2\,z_3^2\,z_4^3\,z_5^3+6\,z_1\,z_2^3\,z_3^2\,z_4^3\,z_5^3\\
&\qquad+6\,z_1^2\,z_2\,z_3^3\,z_4^3\,z_5^3+6\,z_1\,z_2^2\,z_3^3\,z_4^3\,z_5^3+2\,z_1^4\,z_2^3\,z_3^3\,z_4\,z_6+5\,z_1^4\,z_2^3\,z_3^2\,z_4^2\,z_6\\
&\qquad+7\,z_1^4\,z_2^2\,z_3^3\,z_4^2\,z_6+10\,z_1^3\,z_2^3\,z_3^3\,z_4^2\,z_6+3\,z_1^4\,z_2^3\,z_3\,z_4^3\,z_6+12\,z_1^4\,z_2^2\,z_3^2\,z_4^3\,z_6\\
&\qquad+10\,z_1^3\,z_2^3\,z_3^2\,z_4^3\,z_6+3\,z_1^4\,z_2\,z_3^3\,z_4^3\,z_6+10\,z_1^3\,z_2^2\,z_3^3\,z_4^3\,z_6+\ z_1^4\,z_2^3\,z_3^3\,z_5\,z_6\\
&\qquad+7\,z_1^4\,z_2^3\,z_3^2\,z_4\,z_5\,z_6+7\,z_1^4\,z_2^2\,z_3^3\,z_4\,z_5\,z_6+19\,z_1^3\,z_2^3\,z_3^3\,z_4\,z_5\,z_6\\
&\qquad+10\,z_1^4\,z_2^3\,z_3\,z_4^2\,z_5\,z_6+32\,z_1^4\,z_2^2\,z_3^2\,z_4^2\,z_5\,z_6+51\,z_1^3\,z_2^3\,z_3^2\,z_4^2\,z_5\,z_6\\
&\qquad+10\,z_1^4\,z_2\,z_3^3\,z_4^2\,z_5\,z_6+51\,z_1^3\,z_2^2\,z_3^3\,z_4^2\,z_5\,z_6+34\,z_1^2\,z_2^3\,z_3^3\,z_4^2\,z_5\,z_6\\
&\qquad+3\,z_1^4\,z_2^3\,z_4^3\,z_5\,z_6+25\,z_1^4\,z_2^2\,z_3\,z_4^3\,z_5\,z_6+26\,z_1^3\,z_2^3\,z_3\,z_4^3\,z_5\,z_6\\
&\qquad+25\,z_1^4\,z_2\,z_3^2\,z_4^3\,z_5\,z_6+80\,z_1^3\,z_2^2\,z_3^2\,z_4^3\,z_5\,z_6+34\,z_1^2\,z_2^3\,z_3^2\,z_4^3\,z_5\,z_6\\
&\qquad+3\,z_1^4\,z_3^3\,z_4^3\,z_5\,z_6+26\,z_1^3\,z_2\,z_3^3\,z_4^3\,z_5\,z_6+34\,z_1^2\,z_2^2\,z_3^3\,z_4^3\,z_5\,z_6\\
&\qquad+2\,z_1^4\,z_2^3\,z_3^2\,z_5^2\,z_6+6\,z_1^3\,z_2^3\,z_3^3\,z_5^2\,z_6+6\,z_1^4\,z_2^3\,z_3\,z_4\,z_5^2\,z_6\\
&\qquad+13\,z_1^4\,z_2^2\,z_3^2\,z_4\,z_5^2\,z_6+40\,z_1^3\,z_2^3\,z_3^2\,z_4\,z_5^2\,z_6+30\,z_1^3\,z_2^2\,z_3^3\,z_4\,z_5^2\,z_6\\
&\qquad+35\,z_1^2\,z_2^3\,z_3^3\,z_4\,z_5^2\,z_6+4\,z_1^4\,z_2^3\,z_4^2\,z_5^2\,z_6+28\,z_1^4\,z_2^2\,z_3\,z_4^2\,z_5^2\,z_6\\
&\qquad+48\,z_1^3\,z_2^3\,z_3\,z_4^2\,z_5^2\,z_6+20\,z_1^4\,z_2\,z_3^2\,z_4^2\,z_5^2\,z_6+124\,z_1^3\,z_2^2\,z_3^2\,z_4^2\,z_5^2\,z_6\\
&\qquad+91\,z_1^2\,z_2^3\,z_3^2\,z_4^2\,z_5^2\,z_6+35\,z_1^3\,z_2\,z_3^3\,z_4^2\,z_5^2\,z_6+80\,z_1^2\,z_2^2\,z_3^3\,z_4^2\,z_5^2\,z_6\\
&\qquad+30\,z_1\,z_2^3\,z_3^3\,z_4^2\,z_5^2\,z_6+10\,z_1^4\,z_2^2\,z_4^3\,z_5^2\,z_6+12\,z_1^3\,z_2^3\,z_4^3\,z_5^2\,z_6\\
&\qquad+25\,z_1^4\,z_2\,z_3\,z_4^3\,z_5^2\,z_6+84\,z_1^3\,z_2^2\,z_3\,z_4^3\,z_5^2\,z_6+46\,z_1^2\,z_2^3\,z_3\,z_4^3\,z_5^2\,z_6\\
&\qquad+6\,z_1^4\,z_3^2\,z_4^3\,z_5^2\,z_6+74\,z_1^3\,z_2\,z_3^2\,z_4^3\,z_5^2\,z_6+113\,z_1^2\,z_2^2\,z_3^2\,z_4^3\,z_5^2\,z_6\\
&\qquad+30\,z_1\,z_2^3\,z_3^2\,z_4^3\,z_5^2\,z_6+9\,z_1^3\,z_3^3\,z_4^3\,z_5^2\,z_6+39\,z_1^2\,z_2\,z_3^3\,z_4^3\,z_5^2\,z_6\\
&\qquad+30\,z_1\,z_2^2\,z_3^3\,z_4^3\,z_5^2\,z_6+z_1^4\,z_2^3\,z_3\,z_5^3\,z_6+6\,z_1^3\,z_2^3\,z_3^2\,z_5^3\,z_6\\
&\qquad+5\,z_1^2\,z_2^3\,z_3^3\,z_5^3\,z_6+z_1^4\,z_2^3\,z_4\,z_5^3\,z_6+6\,z_1^4\,z_2^2\,z_3\,z_4\,z_5^3\,z_6
\end{aligned}
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\begin{aligned}
&+ 15\, z_1^3\, z_2^3\, z_3\, z_4\, z_5^3\, z_6 + 29\, z_1^3\, z_2^2\, z_3^2\, z_4\, z_5^3\, z_6 + 37\, z_1^2\, z_2^3\, z_3^2\, z_4\, z_5^3\, z_6 \\
&+ 23\, z_1^2\, z_2^2\, z_3^3\, z_4\, z_5^3\, z_6 + 15\, z_1\, z_2^3\, z_3^3\, z_4\, z_5^3\, z_6 + 4\, z_1^4\, z_2^2\, z_4^2\, z_5^3\, z_6 \\
&+ 6\, z_1^3\, z_2^3\, z_4^2\, z_5^3\, z_6 + 10\, z_1^4\, z_2\, z_3\, z_4^2\, z_5^3\, z_6 + 48\, z_1^3\, z_2^2\, z_3\, z_4^2\, z_5^3\, z_6 \\
&+ 39\, z_1^2\, z_2^3\, z_3\, z_4^2\, z_5^3\, z_6 + 35\, z_1^3\, z_2\, z_3^2\, z_4^2\, z_5^3\, z_6 + 86\, z_1^2\, z_2^2\, z_3^2\, z_4^2\, z_5^3\, z_6 \\
&+ 36\, z_1\, z_2^3\, z_3^2\, z_4^2\, z_5^3\, z_6 + 25\, z_1^2\, z_2\, z_3^3\, z_4^2\, z_5^3\, z_6 + 30\, z_1\, z_2^2\, z_3^3\, z_4^2\, z_5^3\, z_6 \\
&+ 6\, z_2^3\, z_3^3\, z_4^2\, z_5^3\, z_6 + 3\, z_1^4\, z_2\, z_4^3\, z_5^3\, z_6 + 12\, z_1^3\, z_2^2\, z_4^3\, z_5^3\, z_6 \\
&+ 9\, z_1^2\, z_2^3\, z_4^3\, z_5^3\, z_6 + 3\, z_1^4\, z_3\, z_4^3\, z_5^3\, z_6 + 26\, z_1^3\, z_2\, z_3\, z_4^3\, z_5^3\, z_6 \\
&+ 46\, z_1^2\, z_2^2\, z_3\, z_4^3\, z_5^3\, z_6 + 17\, z_1\, z_2^3\, z_3\, z_4^3\, z_5^3\, z_6 + 9\, z_1^3\, z_3^2\, z_4^3\, z_5^3\, z_6 \\
&+ 39\, z_1^2\, z_2\, z_3^2\, z_4^3\, z_5^3\, z_6 + 36\, z_1\, z_2^2\, z_3^2\, z_4^3\, z_5^3\, z_6 + 6\, z_2^3\, z_3^2\, z_4^3\, z_5^3\, z_6 \\
&+ 6\, z_1^2\, z_3^3\, z_4^3\, z_5^3\, z_6 + 13\, z_1\, z_2\, z_3^3\, z_4^3\, z_5^3\, z_6 + 6\, z_2^2\, z_3^3\, z_4^3\, z_5^3\, z_6 \\
&+ 3\, z_1^4\, z_2^2\, z_3^3\, z_4\, z_6^2 + 7\, z_1^3\, z_2^3\, z_3^3\, z_4\, z_6^2 + 9\, z_1^4\, z_2^2\, z_3^2\, z_4^2\, z_6^2 \\
&+ 16\, z_1^3\, z_2^3\, z_3^2\, z_4^2\, z_6^2 + 4\, z_1^4\, z_2\, z_3^3\, z_4^2\, z_6^2 + 23\, z_1^3\, z_2^2\, z_3^3\, z_4^2\, z_6^2 \\
&+ 14\, z_1^2\, z_2^3\, z_3^3\, z_4^2\, z_6^2 + 6\, z_1^4\, z_2^2\, z_3\, z_4^3\, z_6^2 + 9\, z_1^3\, z_2^3\, z_3\, z_4^3\, z_6^2 \\
&+ 10\, z_1^4\, z_2\, z_3^2\, z_4^3\, z_6^2 + 33\, z_1^3\, z_2^2\, z_3^2\, z_4^3\, z_6^2 + 14\, z_1^2\, z_2^3\, z_3^2\, z_4^3\, z_6^2 \\
&+ 12\, z_1^3\, z_2\, z_3^3\, z_4^3\, z_6^2 + 14\, z_1^2\, z_2^2\, z_3^3\, z_4^3\, z_6^2 + 2\, z_1^4\, z_2^2\, z_3^3\, z_5\, z_6^2 \\
&+ 6\, z_1^3\, z_2^3\, z_3^3\, z_5\, z_6^2 + 13\, z_1^4\, z_2^2\, z_3^2\, z_4\, z_5\, z_6^2 + 30\, z_1^3\, z_2^3\, z_3^2\, z_4\, z_5\, z_6^2 \\
&+ 6\, z_1^4\, z_2\, z_3^3\, z_4\, z_5\, z_6^2 + 40\, z_1^3\, z_2^2\, z_3^3\, z_4\, z_5\, z_6^2 + 35\, z_1^2\, z_2^3\, z_3^3\, z_4\, z_5\, z_6^2 \\
&+ 20\, z_1^4\, z_2^2\, z_3\, z_4^2\, z_5\, z_6^2 + 35\, z_1^3\, z_2^3\, z_3\, z_4^2\, z_5\, z_6^2 + 28\, z_1^4\, z_2\, z_3^2\, z_4^2\, z_5\, z_6^2 \\
&+ 124\, z_1^3\, z_2^2\, z_3^2\, z_4^2\, z_5\, z_6^2 + 80\, z_1^2\, z_2^3\, z_3^2\, z_4^2\, z_5\, z_6^2 + 4\, z_1^4\, z_3^3\, z_4^2\, z_5\, z_6^2 \\
&+ 48\, z_1^3\, z_2\, z_3^3\, z_4^2\, z_5\, z_6^2 + 91\, z_1^2\, z_2^2\, z_3^3\, z_4^2\, z_5\, z_6^2 + 30\, z_1\, z_2^3\, z_3^3\, z_4^2\, z_5\, z_6^2 \\
&+ 6\, z_1^4\, z_2^2\, z_4^3\, z_5\, z_6^2 + 9\, z_1^3\, z_2^3\, z_4^3\, z_5\, z_6^2 + 25\, z_1^4\, z_2\, z_3\, z_4^3\, z_5\, z_6^2 \\
&+ 74\, z_1^3\, z_2^2\, z_3\, z_4^3\, z_5\, z_6^2 + 39\, z_1^2\, z_2^3\, z_3\, z_4^3\, z_5\, z_6^2 + 10\, z_1^4\, z_3^2\, z_4^3\, z_5\, z_6^2 \\
&+ 84\, z_1^3\, z_2\, z_3^2\, z_4^3\, z_5\, z_6^2 + 113\, z_1^2\, z_2^2\, z_3^2\, z_4^3\, z_5\, z_6^2 + 30\, z_1\, z_2^3\, z_3^2\, z_4^3\, z_5\, z_6^2 \\
&+ 12\, z_1^3\, z_3^3\, z_4^3\, z_5\, z_6^2 + 46\, z_1^2\, z_2\, z_3^3\, z_4^3\, z_5\, z_6^2 + 30\, z_1\, z_2^2\, z_3^3\, z_4^3\, z_5\, z_6^2 \\
&+ 4\, z_1^4\, z_2^2\, z_3^2\, z_5^2\, z_6^2 + 12\, z_1^3\, z_2^3\, z_3^2\, z_5^2\, z_6^2 + 12\, z_1^3\, z_2^2\, z_3^3\, z_5^2\, z_6^2 \\
&+ 18\, z_1^2\, z_2^3\, z_3^3\, z_5^2\, z_6^2 + 13\, z_1^4\, z_2^2\, z_3\, z_4\, z_5^2\, z_6^2 + 29\, z_1^3\, z_2^3\, z_3\, z_4\, z_5^2\, z_6^2 \\
&+ 13\, z_1^4\, z_2\, z_3^2\, z_4\, z_5^2\, z_6^2 + 92\, z_1^3\, z_2^2\, z_3^2\, z_4\, z_5^2\, z_6^2 + 84\, z_1^2\, z_2^3\, z_3^2\, z_4\, z_5^2\, z_6^2 \\
&+ 29\, z_1^3\, z_2\, z_3^3\, z_4\, z_5^2\, z_6^2 + 84\, z_1^2\, z_2^2\, z_3^3\, z_4\, z_5^2\, z_6^2 + 38\, z_1\, z_2^3\, z_3^3\, z_4\, z_5^2\, z_6^2 \\
&+ 9\, z_1^4\, z_2^2\, z_4^2\, z_5^2\, z_6^2 + 15\, z_1^3\, z_2^3\, z_4^2\, z_5^2\, z_6^2 + 32\, z_1^4\, z_2\, z_3\, z_4^2\, z_5^2\, z_6^2 \\
&+ 124\, z_1^3\, z_2^2\, z_3\, z_4^2\, z_5^2\, z_6^2 + 86\, z_1^2\, z_2^3\, z_3\, z_4^2\, z_5^2\, z_6^2 + 9\, z_1^4\, z_3^2\, z_4^2\, z_5^2\, z_6^2 \\
&+ 124\, z_1^3\, z_2\, z_3^2\, z_4^2\, z_5^2\, z_6^2 + 228\, z_1^2\, z_2^2\, z_3^2\, z_4^2\, z_5^2\, z_6^2 + 82\, z_1\, z_2^3\, z_3^2\, z_4^2\, z_5^2\, z_6^2 \\
&+ 15\, z_1^3\, z_3^3\, z_4^2\, z_5^2\, z_6^2 + 86\, z_1^2\, z_2\, z_3^3\, z_4^2\, z_5^2\, z_6^2 + 82\, z_1\, z_2^2\, z_3^3\, z_4^2\, z_5^2\, z_6^2 \\
&+ 12\, z_2^3\, z_3^3\, z_4^2\, z_5^2\, z_6^2 + 12\, z_1^4\, z_2\, z_4^3\, z_5^2\, z_6^2 + 33\, z_1^3\, z_2^2\, z_4^3\, z_5^2\, z_6^2 \\
&+ 21\, z_1^2\, z_2^3\, z_4^3\, z_5^2\, z_6^2 + 12\, z_1^4\, z_3\, z_4^3\, z_5^2\, z_6^2 + 80\, z_1^3\, z_2\, z_3\, z_4^3\, z_5^2\, z_6^2 \\
&+ 113\, z_1^2\, z_2^2\, z_3\, z_4^3\, z_5^2\, z_6^2 + 36\, z_1\, z_2^3\, z_3\, z_4^3\, z_5^2\, z_6^2 + 33\, z_1^3\, z_3^2\, z_4^3\, z_5^2\, z_6^2 \\
&+ 113\, z_1^2\, z_2\, z_3^2\, z_4^3\, z_5^2\, z_6^2 + 84\, z_1\, z_2^2\, z_3^2\, z_4^3\, z_5^2\, z_6^2 + 12\, z_2^3\, z_3^2\, z_4^3\, z_5^2\, z_6^2 \\
&+ 21\, z_1^2\, z_3^3\, z_4^3\, z_5^2\, z_6^2 + 36\, z_1\, z_2\, z_3^3\, z_4^3\, z_5^2\, z_6^2 + 12\, z_2^2\, z_3^3\, z_4^3\, z_5^2\, z_6^2 \\
&+ 2\, z_1^4\, z_2^2\, z_3\, z_5^3\, z_6^2 + 6\, z_1^3\, z_2^3\, z_3\, z_5^3\, z_6^2 + 12\, z_1^3\, z_2^2\, z_3^2\, z_5^3\, z_6^2 \\
&+ 18\, z_1^2\, z_2^3\, z_3^2\, z_5^3\, z_6^2 + 10\, z_1^2\, z_2^2\, z_3^3\, z_5^3\, z_6^2 + 8\, z_1\, z_2^3\, z_3^3\, z_5^3\, z_6^2
\end{aligned}
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\begin{aligned}
&+3\,z_1^4\,z_2^2\,z_4\,z_5^3\,z_6^2+6\,z_1^3\,z_2^3\,z_4\,z_5^3\,z_6^2+7\,z_1^4\,z_2\,z_3\,z_4\,z_5^3\,z_6^2\\
&+40\,z_1^3\,z_2^2\,z_3\,z_4\,z_5^3\,z_6^2+37\,z_1^2\,z_2^3\,z_3\,z_4\,z_5^3\,z_6^2+30\,z_1^3\,z_2\,z_3^2\,z_4\,z_5^3\,z_6^2\\
&+84\,z_1^2\,z_2^2\,z_3^2\,z_4\,z_5^3\,z_6^2+42\,z_1\,z_2^3\,z_3^2\,z_4\,z_5^3\,z_6^2+23\,z_1^2\,z_2\,z_3^3\,z_4\,z_5^3\,z_6^2\\
&+34\,z_1\,z_2^2\,z_3^3\,z_4\,z_5^3\,z_6^2+8\,z_2^3\,z_3^3\,z_4\,z_5^3\,z_6^2+7\,z_1^4\,z_2\,z_4^2\,z_5^3\,z_6^2\\
&+23\,z_1^3\,z_2^2\,z_4^2\,z_5^3\,z_6^2+16\,z_1^2\,z_2^3\,z_4^2\,z_5^3\,z_6^2+5\,z_1^4\,z_3\,z_4^2\,z_5^3\,z_6^2\\
&+51\,z_1^3\,z_2\,z_3\,z_4^2\,z_5^3\,z_6^2+91\,z_1^2\,z_2^2\,z_3\,z_4^2\,z_5^3\,z_6^2+36\,z_1\,z_2^3\,z_3\,z_4^2\,z_5^3\,z_6^2\\
&+16\,z_1^3\,z_3^2\,z_4^2\,z_5^3\,z_6^2+80\,z_1^2\,z_2\,z_3^2\,z_4^2\,z_5^3\,z_6^2+82\,z_1\,z_2^2\,z_3^2\,z_4^2\,z_5^3\,z_6^2\\
&+16\,z_2^3\,z_3^2\,z_4^2\,z_5^3\,z_6^2+11\,z_1^2\,z_3^3\,z_4^2\,z_5^3\,z_6^2+30\,z_1\,z_2\,z_3^3\,z_4^2\,z_5^3\,z_6^2\\
&+16\,z_2^2\,z_3^3\,z_4^2\,z_5^3\,z_6^2+2\,z_1^4\,z_4^3\,z_5^3\,z_6^2+10\,z_1^3\,z_2\,z_4^3\,z_5^3\,z_6^2\\
&+14\,z_1^2\,z_2^2\,z_4^3\,z_5^3\,z_6^2+6\,z_1\,z_2^3\,z_4^3\,z_5^3\,z_6^2+10\,z_1^3\,z_3\,z_4^3\,z_5^3\,z_6^2\\
&+34\,z_1^2\,z_2\,z_3\,z_4^3\,z_5^3\,z_6^2+30\,z_1\,z_2^2\,z_3\,z_4^3\,z_5^3\,z_6^2+6\,z_2^3\,z_3\,z_4^3\,z_5^3\,z_6^2\\
&+14\,z_1^2\,z_3^2\,z_4^3\,z_5^3\,z_6^2+30\,z_1\,z_2\,z_3^2\,z_4^3\,z_5^3\,z_6^2+12\,z_2^2\,z_3^2\,z_4^3\,z_5^3\,z_6^2\\
&+6\,z_1\,z_3^3\,z_4^3\,z_5^3\,z_6^2+6\,z_2\,z_3^3\,z_4^3\,z_5^3\,z_6^2+z_1^4\,z_2\,z_3^3\,z_4\,z_6^3\\
&+6\,z_1^3\,z_2^2\,z_3^3\,z_4\,z_6^3+5\,z_1^2\,z_2^3\,z_3^3\,z_4\,z_6^3+4\,z_1^4\,z_2\,z_3^2\,z_4^2\,z_6^3\\
&+15\,z_1^3\,z_2^2\,z_3^2\,z_4^2\,z_6^3+11\,z_1^2\,z_2^3\,z_3^2\,z_4^2\,z_6^3+6\,z_1^3\,z_2\,z_3^3\,z_4^2\,z_6^3\\
&+16\,z_1^2\,z_2^2\,z_3^3\,z_4^2\,z_6^3+6\,z_1\,z_2^3\,z_3^3\,z_4^2\,z_6^3+3\,z_1^4\,z_2\,z_3\,z_4^3\,z_6^3\\
&+9\,z_1^3\,z_2^2\,z_3\,z_4^3\,z_6^3+6\,z_1^2\,z_2^3\,z_3\,z_4^3\,z_6^3+12\,z_1^3\,z_2\,z_3^2\,z_4^3\,z_6^3\\
&+21\,z_1^2\,z_2^2\,z_3^2\,z_4^3\,z_6^3+6\,z_1\,z_2^3\,z_3^2\,z_4^3\,z_6^3+9\,z_1^2\,z_2\,z_3^3\,z_4^3\,z_6^3\\
&+6\,z_1\,z_2^2\,z_3^3\,z_4^3\,z_6^3+\ z_1^4\,z_2\,z_3^3\,z_5\,z_6^3+6\,z_1^3\,z_2^2\,z_3^3\,z_5\,z_6^3\\
&+5\,z_1^2\,z_2^3\,z_3^3\,z_5\,z_6^3+6\,z_1^4\,z_2\,z_3^2\,z_4\,z_5\,z_6^3+29\,z_1^3\,z_2^2\,z_3^2\,z_4\,z_5\,z_6^3\\
&+23\,z_1^2\,z_2^3\,z_3^2\,z_4\,z_5\,z_6^3+z_1^4\,z_3^3\,z_4\,z_5\,z_6^3+15\,z_1^3\,z_2\,z_3^3\,z_4\,z_5\,z_6^3\\
&+37\,z_1^2\,z_2^2\,z_3^3\,z_4\,z_5\,z_6^3+15\,z_1\,z_2^3\,z_3^3\,z_4\,z_5\,z_6^3+10\,z_1^4\,z_2\,z_3\,z_4^2\,z_5\,z_6^3\\
&+35\,z_1^3\,z_2^2\,z_3\,z_4^2\,z_5\,z_6^3+25\,z_1^2\,z_2^3\,z_3\,z_4^2\,z_5\,z_6^3+4\,z_1^4\,z_3^2\,z_4^2\,z_5\,z_6^3\\
&+48\,z_1^3\,z_2\,z_3^2\,z_4^2\,z_5\,z_6^3+86\,z_1^2\,z_2^2\,z_3^2\,z_4^2\,z_5\,z_6^3+30\,z_1\,z_2^3\,z_3^2\,z_4^2\,z_5\,z_6^3\\
&+6\,z_1^3\,z_3^3\,z_4^2\,z_5\,z_6^3+39\,z_1^2\,z_2\,z_3^3\,z_4^2\,z_5\,z_6^3+36\,z_1\,z_2^2\,z_3^3\,z_4^2\,z_5\,z_6^3\\
&+6\,z_2^3\,z_3^3\,z_4^2\,z_5\,z_6^3+3\,z_1^4\,z_2\,z_4^3\,z_5\,z_6^3+9\,z_1^3\,z_2^2\,z_4^3\,z_5\,z_6^3\\
&+6\,z_1^2\,z_2^3\,z_4^3\,z_5\,z_6^3+3\,z_1^4\,z_3\,z_4^3\,z_5\,z_6^3+26\,z_1^3\,z_2\,z_3\,z_4^3\,z_5\,z_6^3\\
&+39\,z_1^2\,z_2^2\,z_3\,z_4^3\,z_5\,z_6^3+13\,z_1\,z_2^3\,z_3\,z_4^3\,z_5\,z_6^3+12\,z_1^3\,z_3^2\,z_4^3\,z_5\,z_6^3\\
&+46\,z_1^2\,z_2\,z_3^2\,z_4^3\,z_5\,z_6^3+36\,z_1\,z_2^2\,z_3^2\,z_4^3\,z_5\,z_6^3+6\,z_2^3\,z_3^2\,z_4^3\,z_5\,z_6^3\\
&+9\,z_1^2\,z_3^3\,z_4^3\,z_5\,z_6^3+17\,z_1\,z_2\,z_3^3\,z_4^3\,z_5\,z_6^3+6\,z_2^2\,z_3^3\,z_4^3\,z_5\,z_6^3\\
&+2\,z_1^4\,z_2\,z_3^2\,z_5^2\,z_6^3+12\,z_1^3\,z_2^2\,z_3^2\,z_5^2\,z_6^3+10\,z_1^2\,z_2^3\,z_3^2\,z_5^2\,z_6^3\\
&+6\,z_1^3\,z_2\,z_3^3\,z_5^2\,z_6^3+18\,z_1^2\,z_2^2\,z_3^3\,z_5^2\,z_6^3+8\,z_1\,z_2^3\,z_3^3\,z_5^2\,z_6^3\\
&+7\,z_1^4\,z_2\,z_3\,z_4\,z_5^2\,z_6^3+30\,z_1^3\,z_2^2\,z_3\,z_4\,z_5^2\,z_6^3+23\,z_1^2\,z_2^3\,z_3\,z_4\,z_5^2\,z_6^3\\
&+3\,z_1^4\,z_3^2\,z_4\,z_5^2\,z_6^3+40\,z_1^3\,z_2\,z_3^2\,z_4\,z_5^2\,z_6^3+84\,z_1^2\,z_2^2\,z_3^2\,z_4\,z_5^2\,z_6^3\\
&+34\,z_1\,z_2^3\,z_3^2\,z_4\,z_5^2\,z_6^3+6\,z_1^3\,z_3^3\,z_4\,z_5^2\,z_6^3+37\,z_1^2\,z_2\,z_3^3\,z_4\,z_5^2\,z_6^3\\
&+42\,z_1\,z_2^2\,z_3^3\,z_4\,z_5^2\,z_6^3+8\,z_2^3\,z_3^3\,z_4\,z_5^2\,z_6^3+5\,z_1^4\,z_2\,z_4^2\,z_5^2\,z_6^3\\
&+16\,z_1^3\,z_2^2\,z_4^2\,z_5^2\,z_6^3+11\,z_1^2\,z_2^3\,z_4^2\,z_5^2\,z_6^3+7\,z_1^4\,z_3\,z_4^2\,z_5^2\,z_6^3\\
&+51\,z_1^3\,z_2\,z_3\,z_4^2\,z_5^2\,z_6^3+80\,z_1^2\,z_2^2\,z_3\,z_4^2\,z_5^2\,z_6^3+30\,z_1\,z_2^3\,z_3\,z_4^2\,z_5^2\,z_6^3\\
&+23\,z_1^3\,z_3^2\,z_4^2\,z_5^2\,z_6^3+91\,z_1^2\,z_2\,z_3^2\,z_4^2\,z_5^2\,z_6^3+82\,z_1\,z_2^2\,z_3^2\,z_4^2\,z_5^2\,z_6^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 16\, z_2^3 z_3^2 z_4^2 z_5^2 z_6^3 + 16\, z_1^2 z_3^3 z_4^2 z_5^2 z_6^3 + 36\, z_1 z_2 z_3^3 z_4^2 z_5^2 z_6^3 \\
&+ 16\, z_2^2 z_3^3 z_4^2 z_5^2 z_6^3 + 2\, z_1^4 z_4^3 z_5^2 z_6^3 + 10\, z_1^3 z_2 z_4^3 z_5^2 z_6^3 \\
&+ 14\, z_1^2 z_2^2 z_4^3 z_5^2 z_6^3 + 6\, z_1 z_2^3 z_4^3 z_5^2 z_6^3 + 10\, z_1^3 z_3 z_4^3 z_5^2 z_6^3 \\
&+ 34\, z_1^2 z_2 z_3 z_4^3 z_5^2 z_6^3 + 30\, z_1 z_2^2 z_3 z_4^3 z_5^2 z_6^3 + 6\, z_2^3 z_3 z_4^3 z_5^2 z_6^3 \\
&+ 14\, z_1^2 z_3^2 z_4^3 z_5^2 z_6^3 + 30\, z_1 z_2 z_3^2 z_4^3 z_5^2 z_6^3 + 12\, z_2^2 z_3^2 z_4^3 z_5^2 z_6^3 \\
&+ 6\, z_1 z_3^3 z_4^3 z_5^2 z_6^3 + 6\, z_2 z_3^3 z_4^3 z_5^2 z_6^3 + z_1^4 z_2 z_3 z_5^3 z_6^3 \\
&+ 6\, z_1^3 z_2^2 z_3 z_5^3 z_6^3 + 5\, z_1^2 z_2^3 z_3 z_5^3 z_6^3 + 6\, z_1^3 z_2 z_3^2 z_5^3 z_6^3 \\
&+ 18\, z_1^2 z_2^2 z_3^2 z_5^3 z_6^3 + 8\, z_1 z_2^3 z_3^2 z_5^3 z_6^3 + 5\, z_1^2 z_2 z_3^3 z_5^3 z_6^3 \\
&+ 8\, z_1 z_2^2 z_3^3 z_5^3 z_6^3 + 2\, z_1^4 z_2 z_4 z_5^3 z_6^3 + 7\, z_1^3 z_2^2 z_4 z_5^3 z_6^3 \\
&+ 5\, z_1^2 z_2^3 z_4 z_5^3 z_6^3 + 2\, z_1^4 z_3 z_4 z_5^3 z_6^3 + 19\, z_1^3 z_2 z_3 z_4 z_5^3 z_6^3 \\
&+ 35\, z_1^2 z_2^2 z_3 z_4 z_5^3 z_6^3 + 15\, z_1 z_2^3 z_3 z_4 z_5^3 z_6^3 + 7\, z_1^3 z_3^2 z_4 z_5^3 z_6^3 \\
&+ 35\, z_1^2 z_2 z_3^2 z_4 z_5^3 z_6^3 + 38\, z_1 z_2^2 z_3^2 z_4 z_5^3 z_6^3 + 8\, z_2^3 z_3^2 z_4 z_5^3 z_6^3 \\
&+ 5\, z_1^2 z_3^3 z_4 z_5^3 z_6^3 + 15\, z_1 z_2 z_3^3 z_4 z_5^3 z_6^3 + 8\, z_2^2 z_3^3 z_4 z_5^3 z_6^3 \\
&+ 2\, z_1^4 z_4^2 z_5^3 z_6^3 + 10\, z_1^3 z_2 z_4^2 z_5^3 z_6^3 + 14\, z_1^2 z_2^2 z_4^2 z_5^3 z_6^3 \\
&+ 6\, z_1 z_2^3 z_4^2 z_5^3 z_6^3 + 10\, z_1^3 z_3 z_4^2 z_5^3 z_6^3 + 34\, z_1^2 z_2 z_3 z_4^2 z_5^3 z_6^3 \\
&+ 30\, z_1 z_2^2 z_3 z_4^2 z_5^3 z_6^3 + 6\, z_2^3 z_3 z_4^2 z_5^3 z_6^3 + 14\, z_1^2 z_3^2 z_4^2 z_5^3 z_6^3 \\
&+ 30\, z_1 z_2 z_3^2 z_4^2 z_5^3 z_6^3 + 12\, z_2^2 z_3^2 z_4^2 z_5^3 z_6^3 + 6\, z_1 z_3^3 z_4^2 z_5^3 z_6^3 + 6\, z_2 z_3^3 z_4^2 z_5^3 z_6^3 \,.
\end{aligned}
$$

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